Some questions related to the Analyst's TST and a conjecture of Carleson

Michele Villa

Doctor of Philosophy
University of Edinburgh

Declaration

I declare that the thesis has been composed by myself and that the work has not be submitted for any other degree or professional qualification. I confirm that the work submitted is my own, except where work which has formed part of jointly-authored publications has been included. My contribution and those of the other authors to this work have been explicitly indicated below. I confirm that appropriate credit has been given within this thesis where reference has been made to the work of others.

(Michele Villa)

Abstract

This thesis is constituted by two independent parts. In the first part we discuss several problems which relate to the Analyst's Travelling Salesman Theorem of Peter Jones and the theories of uniformly rectifiable sets and quasiminimal sets of David and Semmes. To say more, this first part splits naturally into two chapters, which are nevertheless connected. In a first chapter, we present a joint work with Jonas Azzam which shows that, in some sense, there exists a travelling salesman theorem for each property that characterises uniformly rectifiable sets. In a second chapter, we address the question of what type of geometric object can play the role of a curve in a higher dimensional setting, at least from the point of view of an Analyst's TST.

The second part of this thesis is dedicated to presenting a joint work with Benjamin Jaye and Xavier Tolsa which settles a longstanding conjecture of Carleson, known in the field as Carleson ϵ^2 conjecture, in a positive sense.

Lay Summary

This thesis is concerned with studying geometrical properties of subsets of Euclidean space. Imagine we collected a set of data which is dependent on some variable x, and imagine we plotted them as points, so to obtain a typical graph with an x-axis and a y-axis. We print the plot, and we see that all the data cluster along a line; from this *geometric* information, we conclude something about the dependence of what we observed on x. But imagine that all of a sudden we become blind[1]. How to check, now, if the points are scattered around the plot in a random fashion, or if they cluster around a line, or perhaps around a curve? A possible answer is to train an ant to go and check this for us. But there are many ways to test whether the points 'have a shape' or not; so, for example, we could measure the distance between each point and a best fitting line, or perhaps two best lines, or, we could also see if the points lie symmetric with respect to each other.

For the moment, suppose we want to measure the distance between points and a best fitting line. Even if this measurement would return small values, it could be that, zooming into our plot around a certain area, the points start looking messy, and not clustering around a line at all. Our ants should go and check at all scales and locations in the plot. When they return with the measurements, we will know a great deal about the geometry of the collected data.

Multiscale measurements of geometric properties of sets (such as the property of lying around a line or being symmetric, or being *topologically stable*) is the central theme of this thesis. We will present several results that relate these measurements to fundamental geometric notions, such as *rectifiability* or *uniform rectifiability*; notions which in turn dictate the type of mathematical analysis that can be done on them, if any. An analogy to our initial situation: if the points are clustered around a line, we can hope to draw some conclusion; if they are scattered around without apparent shape, we should hope a little less.

[1] We would be in a similar situation if instead of suddenly becoming blind, we found out that our observations depend on millions of variables.

Acknowledgements

First of all, I thank Jonas Azzam, my supervisor. He taught me a lot of mathematics, and, just as importantly, he provided me with a source of enthusiasm when mine was almost gone. Numerous times it happened that I entered his office pale and haggard, just to come out an hour later tanned and chipper as after a holiday. I would also like to thank Xavier Tolsa. During my visit to UAB I learnt a great deal from the experience of working with him. I should also thank him and Benjamin Jaye for the very pleasant collaboration on a paper which forms a chapter of this thesis. During my time at UAB, I enjoyed (and benefited from) many hours of mathematical discussion with Joan Mateu and Damian Dabrowski. Finally, I thank Tony Carbery, Jim Wright, Dugald Duncan and Isabelle Hanlon for creating an amazing learning environment with the MIGSAA CDT.

I am grateful to my family, Mariella, Matteo, Sofia and Tommaso. Their support and love is a fundamental invariant of my life; any step I make, just like the first one, finds solid ground in them. The same has been true, in more recent years, for Isabella. Thank you for your tender love, intelligence and guidance. I want to mention all my relatives (they are many!): Giovanna, Paolo, Chiara, Pietro, Rosanna, Paolo, Francesco, Davide, Marco and Cinzia. Their generous and continuous help throughout the bachelor years has been a necessary condition for getting here.

I would also like to thank Justin, Stefania and Victor, and many other PhD students at Edinburgh, for making me laugh a lot (most times). Finally, last but not least, I wish to thank Andrea, for consistently being a *sonsino*.

Contents

CHAPTER 1

Introduction

In this introduction we set the stage for the research work that this thesis presents. In other words, we try to describe the mathematical context within which the results we are going to describe find meaning.

1. Origins

This context can be summarised with the phrase 'quantitative rectifiability': a loose set of techniques and problems which started to form in the late eighties and nineties. We will not try to trace in details this 'formation process': for the purposes of this introduction a brief sketch will do. It all started[1] with the following problem. Let $\Omega \subset \mathbb{C}$ be open. A subset $E \subset \Omega$ is said to be removable if every bounded function analytic on $\Omega \setminus E$ has an analytic extension to Ω. The question is: can we understand their geometry? More precisely: can we give a characterisation of removable sets in geometric terms? This question was formulated more than a hundred years ago and came to be known as the Painlevé problem. It soon became clear that the critical dimension[2] for the problem is one: sets which have dimension smaller than one are removable, and the ones whose dimension is larger than one are non-removable. Now, it is a fundamental dichotomy in geometric measure theory that any one-dimensional set with *finite* 1-measure in Euclidean space can be decomposed into two subsets, one which is 1-*rectifiable* and another which is *purely* 1-*unrectifiable* (see [**Mat95**, Theorem 15.5]). We recall their definition[3]. A set $E \subset \mathbb{R}^2$ is called 1-rectifiable if there are Lipschitz maps $f_i : \mathbb{R} \to \mathbb{R}^2$, $i = 1, 2, \ldots$, such that

$$\mathcal{H}^1\Big(E \setminus \textstyle\bigcup_i f_i(\mathbb{R})\Big) = 0. \tag{1.1}$$

(Here $\mathcal{H}^d$ is the d-dimensional Hausdorff measure. See Appendix, Subsection 1.2 for definitions). A set $E \subset \mathbb{R}^2$ is purely 1-unrectifiable if, for any Lipschitz map f as above, $\mathcal{H}^1(E \cap f(\mathbb{R})) = 0$.

A special case of Painlevé problem was re-formulated by Vitushkin in the fifties: what came to be known as Vitushkin conjecture asserts that a set E with *finite* Hausdorff measure is removable if and only if it is purely 1-unrectifiable[4]. In the eighties, it became understood that there is a close relationship between the removability (or non removability) of a set E and the *Cauchy integral operator* on E. Given a subset E of the the plane and a function f, the Cauchy integral operator on E, which we denote by $\mathscr{C}_E$, is formally defined as

$$\mathscr{C}_E f(x) := \int \frac{f(y)}{x-y} \, d\mathcal{H}^1|_E(y).$$

Roughly, the relationship is that E is non-removable if and only if $\mathscr{C}_E$ is bounded[5] as an operator $L^2(E) \to L^2(E)$ (see [**C90, C90b**]).

Let us go back to the story of the Painlevé problem. The first part of the conjecture followed as a consequence of Calderón's result on the boundedness of the Cauchy integral operator on Γ, where Γ is a Lipschitz curve with small slope ([**Ca77**]).

A proof of the reverse implication, that pure unrectifiability implies removability didn't materialise until much more recently. This has been due to the difficulty in deducing geometric

[1] This view is perhaps simplistic, but not erroneous.

[2] For dimension, here we mean Hausdorff dimension. See [**Mat95**, Definition 4.8] for a definition of this. See [**To14**, Section 1.6] for why one it the critical dimension.

[3] Most notation in the introduction should be self-explanatory; however, for convenience, we recalled almost all notation in the Appendix (see (5))

[4] We are altogether brushing analytic capacity under the rug .

[5] For what is meant by $L^2(E)$ boundedness, see for example [**DS93**], Part I, Chapter 1, Section 1.2, or [**Mat95**], page 278.

information on a set E from analytical information on the Cauchy integral on E. In the nineties, Melnikov ([**Mel95**]) found the right tool to connect information about the Cauchy integral operator on sets to their geometry: the Menger curvature. This quickly lead to a proof of the remaining direction of the Vitushkin conjecture, first, in a special case, by Mattila, Melnikov and Verdera ([**MMV97**]), and subsequently, in the general case, by David ([**Dav98**]).

The Painlevé problem for sets with infinite measure was given a solution by Tolsa in the early two-thousands ([**To03**]). Tolsa's solution was in terms of Menger curvature; indeed a characterisation in terms of rectifiability was previously shown to be false by Mattila ([**Mat96**]).

2. Developments

In this section, we will try to convince the reader that, while the initial question of Painlevé played the role of first seed, it's solution is one of the many spectacular branches of a robust tree. We will focus on two branches.

2.1. Uniform rectifiability. As was mentioned above, to understand the connection between removability and geometry of a set E, one needed to understand the connection between the behaviour of the Cauchy integral operator on E and the geometry of E. This is a question that has it's own independent interest; it is natural to give it more space and breath: in the two monographs [**DS91**] and [**DS93**], David and Semmes addressed the following: given a set $E \subset \mathbb{R}^n$, equipped with the measure $\mathcal{H}^d$, which is assumed to be locally finite, what *geometric* condition should one put on E, so that the Cauchy transform, or a wider class of singular integral operators (SIOs), are bounded on $L^2(E)$? Rectifiability alone does not suffice (see [**DS93**, I.1], Section 1.2), essentially because it is a qualitative notion, while that a SIO is $L^2(E)$ bounded is a quantitative notion.

Throughout their work, David and Semmes made the technical assumption of *Ahlfors regularity*. A set E is said to be Ahlfors d-regular with constant c_0, if for all $x \in E$ and $0 < r < \operatorname{diam}(E)$, it holds that

$$\mathsf{c}_0^{-1}\, r^d \leq \mathcal{H}^d(B(x,r) \cap E) \leq \mathsf{c}_0\, r^d. \tag{1.2}$$

Ahlfors regularity is a useful assumption to make, as it gives a clear relationship between the diameter of a set and its size in a scale-invariant and quantitative way; moreover, $\mathcal{H}^d|_E$ will be a doubling measure, hence $(E, \mathcal{H}^d|_E)$ will be a space of homogeneous type (as in [**CW77**]), where many of the tools from the Euclidean Calderón-Zygmund theory apply. It should be noted, however, that Ahlfors regularity is not, strictly speaking, a regularity assumption, in the sense of constraining the geometry of E. The well known Garnett's example (see [**DS93**], page 9), also known as the four corner Cantor set, is Ahlfors 1-regular but also purely 1-unrectifiable.

But lets go back to the question of David and Semmes: we are looking for a geometric condition which guarantees L^2 boundedness of plenty of singular integral operators; this should involve rectifiability but also be quantitative. The main achievement of David and Semmes was to *single out* the right notion of quantitative rectifiability and to provide a multitude of (non-trivially) equivalent geometric conditions that guarantee this notion, and thus that guarantee the L^2 boundedness of plenty of singular integral operators. This notion was termed by David and Semmes *uniform rectifiability*. The fundamental property of uniformly rectifiable sets is the following: suppose that we are looking at E within a ball centered on E and with a certain radius $r > 0$. Then, for any such ball, we will have that at least ten percent, say, of the portion of E that we are able to see is covered by *just one* Lipshitz image. This proportion, ten percent, is *uniform* with respect to the scale and location of balls. Hence the name. Let us now give the precise definition.

Definition 1.1. An Ahlfors d-regular set $E \subset \mathbb{R}^n$ is uniformly d-rectifiable if and only if there exists positive numbers M and θ such that for each $x \in E$ and each $0 < r < \operatorname{diam}(E)$ there is a Lipschitz map $f : \mathbb{R}^d \to \mathbb{R}^n$ such that

$$\mathcal{H}^d(E \cap B(x,r) \cap f(B(0,r))) \geq \theta r^d.$$

As mentioned above, there are many other equivalent definitiona of uniform rectifiability; we will see several of them in the next chapter.

Another way of looking at David and Semmes work is through *analogy*, although this point of view has perhaps only emerged *a posteriori*, or as an ulterior, albeit very powerful, connection. Consider $f:\mathbb{R}\to\mathbb{R}$ in $L^2(\mathbb{R})$. Then (see [SM93]) f is locally absolute continuous and $f'\in L^2$ if and only if $\int_0^\infty\int_{\mathbb{R}} t^{-2}|f(x+t)+f(x-t)-2f(x)|^2\,dx\frac{dt}{t}<\infty$. The quantity $t^{-1}(f(x+t)+f(x-t)-2f(x))$ can be seen as a measurement of how far f is from being affine. A result in the same spirit is the following theorem of Dorronsoro ([D85]). Let $f\in W^{1,2}(\mathbb{R}^d)$ and define $\Omega_f(x,t):=\left(\inf_A t^{-d}\int_{B(x,t)} t^{-2}\,(f(y)-A(y))^2\,dy\right)^{\frac{1}{2}}$, where the infimum is taken over all affine functions A. Then

$$\int_{\mathbb{R}^d}\int_0^\infty \Omega_f(x,t)^2\,\frac{dt}{t}dx\sim\|\nabla f\|_2^2. \tag{1.3}$$

Thus a function has a square integrable derivative if and only if it 'becomes affine' sufficiently fast as we zoom in around a point. Since rectifiable sets are composed of Lipschitz images, we know that they have tangent almost everywhere. It is natural therefore to wonder if it is possible to measure in a quantitative way how fast rectifiable sets 'become flat', and if one can give a quantitative estimate of the type (1.3), which, rather than giving information on the differentiability properties of functions, it tells us about rectifiability properties of sets. As first step one needs an analogue of the Ω_f coefficients. Take a subset $E\subset\mathbb{R}^n$ and let

$$\beta_{E,p}^d(x,t):=\inf_L\left(\frac{1}{t^n}\int_{B\cap E}\left(\frac{\mathrm{dist}(y,L)}{t}\right)^p d\mathcal{H}^d(y)\right)^{\frac{1}{p}}, \tag{1.4}$$

Note the analogy; in the case of functions, we look at their square difference to the best approximating affine map, while in the case of sets, we look at the square difference to the best approximating plane. David and Semmes proved the following.

THEOREM 1.2. *An Ahlfors regular set E is uniformly d-rectifiable if and only if there exists a constant C such that for all $x\in E$ and $0<r<\mathrm{diam}(E)$, we have that*

$$\int_{E\cap B(x,r)}\int_0^r \beta_{E,p}^d(y,s)^2\,\frac{ds}{s}\,d\mathcal{H}^d(y)\le C\,r^d. \tag{1.5}$$

Thus if a set 'becomes flat' sufficiently fast, then not only is it rectifiable, it is uniformly rectifiable, which is a stronger property; to stretch the analogy, one may think about the difference between the existence of pointwise derivative and the gradient being in L^2.

The estimate (1.5) tells us that the measure $\beta_{E,p}^d(y,s)^2\,ds/s\,d\mathcal{H}^d(y)$ is a *Carleson measure.* Historically, Carleson measures have played an important role in harmonic analysis (see for example [SM93], Chapter 2). That $\beta_{E,p}^d(y,s)^2\,ds/s\,d\mathcal{H}^d(y)$ is a Carleson measure roughly means that the set of pairs $(x,r)\in E\times(0,\mathrm{diam}(E))$ where the β coefficients are large, must be small.

Now, many computations in harmonic analysis are more easily done by using dyadic-type decompositions of the space under study; in this case, our space is simply E with the Euclidean metric. Let $\mathcal{D}$ denote one such decomposition of E (see the Appendix, Theorem 5.2 or Theorem 5.3 below for precise definitions and notation). To each pair (x,r) there corresponds a cube, roughly located near x and with side length comparable to r. Thus, to say that $\beta_{E,p}^d(y,s)^2\,ds/s\,d\mathcal{H}^d(y)$ is a Carleson measure is equivalent to say that the family of cubes Q from $\mathcal{D}$ where the β coefficients are larger than a given threshold is small family. This motivates the following definition. We say that a family of cubes $\mathcal{F}$ satisfies a *Carleson packing condition* if there is a constant C so that for all $R\in\mathcal{D}$,

$$\sum_{\substack{Q\in\mathcal{F}\\ Q\subseteq R}}\ell(Q)^d\le C\ell(R)^d. \tag{1.6}$$

We will call a family $\mathcal{F}$ satisfying (1.6) a *Carleson family.* Let us now mention a couple more characterisations of uniformly d-rectifiable sets given by David and Semmes. They are in the same vein of Theorem 1.2, except they are given in terms of Carleson families.

Given two closed sets E and F, and B a ball, we denote

$$d_B(E,F)=\frac{2}{\mathrm{diam}\,B}\max\left\{\sup_{y\in E\cap B}\mathrm{dist}(y,F),\ \sup_{y\in F\cap B}\mathrm{dist}(y,E)\right\}. \tag{1.7}$$

For a constant $\mathsf{C}_0 > 0$, and $\epsilon > 0$, let

$$\text{BWGL}(\mathsf{C}_0, \epsilon) = \{Q \in \mathcal{D} | \; d_{\mathsf{C}_0 B_Q}(E, P) \geq \epsilon \quad \text{for all } d\text{-planes } P\}. \tag{1.8}$$

The family $\text{BWGL}(\mathsf{C}_0, \epsilon)$ (here BWGL stands for *bilateral weak geometric lemma*) is then the set of cubes where E is not very well behaved, that is, where E can't be approximeted well (bilaterally) by unions of planes. Following the rationale mentioned above, one would guess that $\text{BWGL}(\mathsf{C}_0, \epsilon)$ should be small, in the sense of satisfying an estimate like (1.6), whenever one is dealing with a uniformly rectifiable set E. This is precisely the case.

THEOREM 1.3 ([**DS93**], Part I, Theorem 2.4). *An Ahlfors d-regular set E is uniformly d-rectifiable if and only if for every $\mathsf{C}_0 \geq 1$ there is $\epsilon > 0$ sufficiently small so that* $\text{BLWG}(\mathsf{C}_0, \epsilon)$ *satisfies a Carleson packing condition (with constant depending on ϵ).*

Another important classification from [**DS93**] is that involving *bilateral approximation by unions of planes* (BAUP): for $R \in \mathcal{D}$ and $\epsilon > 0$, let

$$\text{BAUP}(\mathsf{C}_0, \epsilon) = \{Q \in \mathcal{D} | \; d_{\mathsf{C}_0 B_Q}(E, U) \geq \epsilon, \; U \text{ is a union of } d\text{-planes}\}. \tag{1.9}$$

We have the following theorem.

THEOREM 1.4 ([**DS93**], Part II, Proposition 3.18). *An Ahlfors d-regular set E is uniformly d-rectifiable if and only if* $\text{BAUP}(\mathsf{C}_0, \epsilon)$ *satisfies a Carleson packing condition for each $\mathsf{C}_0 > 1$ and $\epsilon > 0$ small enough (depending on C_0).*

Theorem 1.4 was a key tool in [**HLMN17**] in showing that the weak-A_∞ condition for harmonic measure implies uniform rectifiability, and also was key in Nazarov, Tolsa, and Volberg's solution to David and Semmes' conjecture in codimension 1 [**NTV14**].

These applications suggests that the study of uniformly rectifiable sets, in terms of characterisations such as Theorem 1.3 or 1.4, and, more generally, understanding the geometry of sets in quantitative terms, have its own importance and relevance. Indeed, in the past ten years, there has been a flurry of activity in this direction, where the interest in questions concerning quantitative rectifiability has become independent to the original problems about removability and SIOs.

In turn, techniques and results so obtained, found application in altogether different areas of analysis. For example, a result of Azzam and Tolsa ([**AT15**]) on the connection between the β coefficients and rectifiability found application in free boundary problems ([**DMSV16**] , [**EE19**]). Or, equally interestingly, ideas developed in the quantitative rectifiability context found use in the solution to old problems in complex analysis and harmonic measure; a good example of this is Chapter 4 of this thesis.

2.2. The Analyst's Travelling Salesman Theorem. We now take a look at a second bough of the 'quantitative rectifiability' tree, or context.

We begin by recalling the Travelling Salesman Problem (TSP). Given a finite set of points, the TSP asks to find the shortest path that goes through each one of them. In the late eighties, Jones posed the following variant of the TSP: given a general set E in the plane, can we find the length of the shortest curve (up to a constant multiple) passing through it? This makes sense whenever we can actually cover E with a rectifiable curve. Hence a restatement of Jones' question is: give a characterisation of subsets E of rectifiable curves in the plane. This problem came to be known as the Analyst's TSP. Note that if we know from the start that E is contained in a line L, then we are immediately done: the shortest curve Γ will be the appropriate line segment, and it's length will be exactly equal to the diameter of E. However, if E presents some curvature, then the length of the covering curve will necessarily increase; to quantify this increment, one needs to quantify how much E deviates from being a line, that is, we need a measurement of its curvature. Hence it makes sense to measure how much E deviates from lines at all points and scales: so Jones defined what came to be known as the Jones β numbers; for a dyadic square Q, set

$$\beta_{E,\infty}(Q) := \inf_{L \text{ a line}} \sup_{z \in E \cap 3Q} \frac{\text{dist}(z, L)}{\ell(Q)}, \tag{1.10}$$

where $\ell(Q)$ denotes the side length of Q, the infimum is taken over all (affine) lines in $\mathbb{C}$ and $\text{dist}(z, L)$ is the usual Hausdorff distance given by $\inf_{l \in L} |z - l|$. Note that this quantity measures exactly what was mentioned above, i.e. how much E deviates from a line (Q determines the scale and location at which we make the measurement). Jones in [**J90**] then proved the following.

THEOREM 1.5. *E is a subset of a curve Γ and Γ is rectifiable if and only if*

$$\beta_\infty^2(E) := \sum_{Q \text{ dyadic}} \beta_{E,\infty}^2(Q)\ell(Q) < \infty, \tag{1.11}$$

where the sum is taken over all dyadic cubes. Moreover, this statement is quantitative, that is, let Γ_0 denotes the shortest curve containing E; there exists a universal constant C_0 such that for all E we have

$$C_0^{-1} \leq \frac{\text{diameter}(E) + \beta_\infty^2(E)}{\mathcal{H}^1(\Gamma_0)} \leq C_0. \tag{1.12}$$

Let us make a couple of remarks. First, note that Jones' result predates Theorem 1.2 of David and Semmes, and they, most likely, conceived their result under the influence of Jones's theorem[6]. Secondly, we want to mention that Jones's result was perhaps one of the first that can be framed within the quantitative rectifiability context. Indeed, on one hand, the motivating problem is, once again, the question of Painlevé; Jones was trying to obtain results relating the L^2 boundedness of the Cauchy transform on E to the geometry of E when he proved his theorem. On the other hand, the result itself is a quantitative result, an estimate, which give information about the rectifiability properties of a set.

To complete the TSP story, the same characterisation was later generalised to curves in arbitrary euclidean space by K. Okikiolu in [**Oki**] and to curves in Hilbert space by R. Schul in [**Sch07**]. The question on whether a similar theorem could be proved for higher dimensional sets was completely open until very recently. There are two reasons for this. The first, technical in nature, is that in any situation where one is dealing with sets of dimension larger than one, Jones' $\beta_{E,\infty}$ coefficients (as in (1.10)) become rather useless: in his PhD thesis, X. Fang constructed a Lipschitz graph G with $\beta_\infty^2(G) = \infty$, see [**Fa90**]. Also the L^p version of β coefficients devised by David and Semmes cannot work: with E Ahlfors d-regular, the definition in (1.4) makes sense. However, Jones's theorem did not assume this; indeed, E need not a priori have finite d-dimensional measure, let alone be d-regular. Thus, a new type of β coefficient was needed. The second issue concerning a higher dimensional TST is the more fundamental one: if in the plane we characterise subsets of rectifiable curves, subsets of what type of geometric object do we want to consider now? Or, in other words, what sort of sets in $\mathbb{R}^n$ should play the role that curves played in the plane? One could legitimately think about, for example, topological spheres; see Figure 2.2 for why this would not be a good candidate.

Notwithstanding these difficulties, a few of years ago, J. Azzam and R. Schul proved in [**AS18**] a version of Jones' theorem for sets of dimension larger than one in Euclidean space. To do so, they introduced the following variant of the L^p-type β coefficients. They put

$$\beta_E^{d,p}(x,r) = \inf_L \left(\frac{1}{r^d} \int_0^1 \mathcal{H}_\infty^d(\{y \in B(x,r) \cap E \,|\, \text{dist}(y,L) > tr\}) t^{p-1}\, dt \right)^{\frac{1}{p}}, \tag{1.13}$$

where the infimum is taken over all affine d-planes L in $\mathbb{R}^n$. The integral on the right hand side of (1.13) is a Choquet integral. See the Appendix for a definition of $\mathcal{H}_\infty^d$. For the moment, let us just recall that $\mathcal{H}_\infty^d$ is somewhat similar to $\mathcal{H}^d$; however, it is not a measure. Azzam and Schul chose $\mathcal{H}_\infty^d$ to define their β coefficients, because $\mathcal{H}_\infty^d$ *does not blowup.* Indeed, for any set E, we always have that $\mathcal{H}_\infty^d(E) \lesssim \text{diam}(E)^d$, even when the Hausdorff dimension of E is larger than d. To deal with the second difficulty, Azzam and Schul decided for a slightly different approach to that of Jones. They chose to focus on obtaining a quantitative result of type (1.12) for a set E lying in $\mathbb{R}^n$ by imposing a certain size condition directly on E. This size condition is the following:

DEFINITION 1.6. We say that a set $E \subset \mathbb{R}^n$ is *lower content d-regular* with constant $\mathsf{c}_1 < 1$, or lower content (d, c_1)-regular, if

$$\mathcal{H}_\infty^d(B(x,r) \cap E) \geq \mathsf{c}_1 r^d \tag{1.14}$$

for all $(x,r) \in E \times (0, \text{diam}(E))$.

[6]In fact, both the coefficients in (1.4) and in (1.10) are called Jones β numbers.

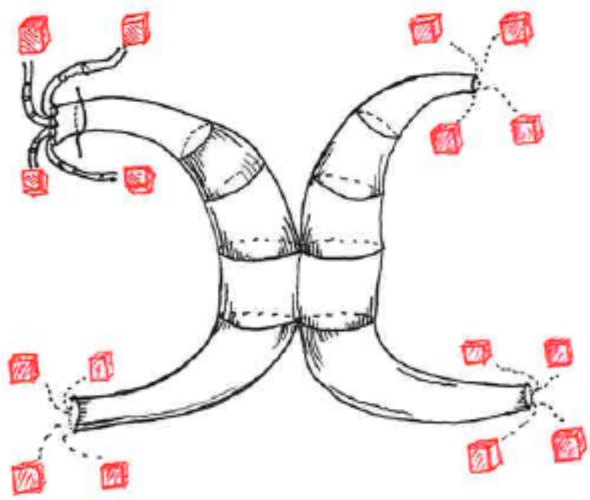

FIGURE 1. Given the 2-dimensional 8-corner Cantor set in $\mathbb{R}^3$, one can construct a 2-dimensional surface with finite measure, so that the closure of this surface will contain the Cantor set and will be homeomorphic to the 2-sphere.

Note in particular that a curve is lower content 1-regular; hence E is, in this respect, 'similar' to a curve (it is at the same time true that there are no assumptions on E in Jones' theorem: some questions remain unanswered). With these two adjustments, Azzam and Schul proved an estimate of the type (1.12), with a difference, however: rather than having only the d-dimensional Hausdorff measure of E at the denominator (as in (1.12)), they had $\mathcal{H}^d(E)$ *plus* a term, BWGL(E), which is given by $\sum_{Q \in \mathrm{BWGL}} \ell(Q)^d$ (recall the definition of BWGL in (1.8))[7], which here quantifies the amount of 'holes' present in E; that is, they proved an estimate like

$$C^{-1} \leq \frac{\text{diameter}(E)^d + \beta_E^{d,p}(E)^2}{\mathcal{H}^d(E) + \mathrm{BWGL}(E)} \leq C, \tag{1.15}$$

where $\beta^{d,p}(E)^2$ is defined as in (1.11), but using the content β numbers. The presence of BWGL(E) is somewhat natural: in Jones's theorem, we had at the denominator the length of the minimal curve covering E; now, a curve has no holes. However, E may very well be quite broken (even while being lower content regular). Thus, if we imagine our set E being covered by 'a higher dimensional curve' Γ, we would have $\mathcal{H}^d(\Gamma) \sim \mathcal{H}^d(E) + \mathrm{BWGL}(E)$, where $\mathrm{BWGL}(E) \sim \mathcal{H}^d(\Gamma \setminus E)$.

3. Problems and results

We now have a little bit of context: we look for questions that transpire from it.

3.1. Quantitative comparisons without Ahlfors regularity. The properties of being bilaterally well approximated by planes (BWGL) and by union of planes (BAUP) are just two of the *quantitative properties* that David and Semmes showed to be strongly connected to uniform rectifiability, in the sense of Carleson packing conditions mentioned above. We will describe some of these properties in Chapter 2. For the moment, let us just say that all the characterisations involved, read roughly as follows: given a quantitative property, an Ahlfors regular set is uniformly rectifiable if and only if the family of cubes where such property fails satisfies a Carleson packing condition. Hence, sums such as $\mathrm{BWGL(E)} = \sum_{\mathrm{Q} \in \mathrm{BWGL}} \ell(\mathrm{Q})^{\mathrm{d}}$, or $\mathrm{BAUP(E)} = \sum_{\mathrm{Q} \in \mathrm{BAUP}} \ell(\mathrm{Q})^{\mathrm{d}}$, or over any other quantitative property (within a certain group that we will specify in Chapter 2) all add up (at most) to the same constant; in other words, they measure the same thing. This, whenever we are dealing with Ahlfors regular sets. Can the same be said for more general sets? That is, are these various properties still strongly linked,

[7]At this point we are purposedly ignoring the parameters C_0 and ϵ.

even when we are looking at, for example, a lower content regular set? To be more precise, does an estimate like (1.15) hold, if we substitute BWGL(E) with, say BAUP? We answer positively to this question with Theorem 2.2, Chapter 2, which was proven jointly with Jonas Azzam.

THEOREM 1.7. *Let $E \subseteq \mathbb{R}^n$ be a lower content (d, c_1)-regular set with Christ-David cubes $\mathcal{D}$. For $R \in \mathcal{D}$, define*

$$\mathrm{BAUP}(R) := \mathrm{BAUP}(R, \mathsf{C}_0, \epsilon) = \sum_{\substack{Q \subseteq R \\ Q \in \mathrm{BAUP}(\mathsf{C}_0, \epsilon)}} \ell(Q)^d.$$

For all $R \in \mathcal{D}$, $\mathsf{C}_0 > 1$, and $\epsilon > 0$ small enough depending on C_0 and c,

$$\mathscr{H}^d(R) + \mathrm{BAUP}(R, \mathsf{C}_0, \epsilon) \sim_{\mathsf{C}_0, \epsilon, \mathsf{c}_1} \beta_E(R). \tag{1.16}$$

This in particular, implies that $\sum_{Q \in \mathrm{BWGL}} \ell(Q)^d \sim \sum_{Q \in \mathrm{BAUP}} \ell(Q)^d$; the two sums add up to the same constant, even when they do not satisfy a Carleson packing condition.

To prove this result (and a few more), we will need to construct a coronisation of lower content regular sets by Ahlfors regular sets, which, we think, may be of independent interest.

3.2. Topologically stable surfaces as higher dimensional curves. The story of the Analyst's TST has left an issue open: what to use as curves in higher dimensional settings? Another way to formulate this this question is: what conditions on a set E guarantee an estimate of the type (1.12) *without* the term BWGL(E)? To answer this, we put on E a *topological condition* (firstly appeared in [**Dav04**]) which guarantees precisely this: an estimate of the type (1.12), or, in other words, (1.15) without the BWGL(E) term. We will define this condition further on, see Chapter 3, Subsection 1.3. For the moment, call a set E satisfying such a condition *topologically stable*. A corollary of the main result of Chapter 3 and of the main result of [**AS18**] (see Theorem 2.1, Chapter 2), is the following theorem.

THEOREM 1.8. *Let $E \subset \mathbb{R}^n$ be topologically stable. Then, for $\mathsf{A} \geq 1$ sufficiently large, we have*[8]

$$\mathrm{diam}(Q_0)^d + \sum_{\substack{Q \in \mathcal{D} \\ Q \subset Q_0}} \beta_E^{p,d}(\mathsf{A}Q)^2 \ell(Q)^d \sim \mathcal{H}^d(Q_0), \tag{1.17}$$

where $Q_0 \in \mathcal{D}$ and where the constant behind the symbol $\sim$ depends on d, n, the parameters coming from the topological condition, and the parameters behind the constants appearing in Theorem 2.1, Chapter 2.

Towards the end of Chapter 3 we give an application of this result to a question on the Hausdorff dimension of *uniformly non-flat* sets.

3.3. Carleson's ϵ^2 conjecture. The second part of this thesis (Chapter 4) contains a proof of the so called Carleson ϵ^2 conjecture, jointly written with Benajmin Jaye and Xavier Tolsa. This is a longstanding conjecture which was formulated by Carleson in the eighties. Although this is prior to the work of Jones and David and Semmes, the type of question and the techniques used to answer fully belong to the quantitative rectifiability context.

To formulate the problem we need to introduce some notation. Let Ω^+ be a proper open set in $\mathbb{R}^2$, and set $\Gamma = \partial\Omega^+$ and $\Omega^- = \mathbb{R}^2 \setminus \overline{\Omega^+}$. For $x \in \mathbb{R}^2$ and $r > 0$, denote by $I^+(x,r)$ and $I^-(x,r)$ the longest open arcs of the circumference $\partial B(x,r)$ contained in Ω^+ and Ω^-, respectively (they may be empty). Then we define

$$\epsilon(x,r) = \frac{1}{r} \max\left(\left|\pi r - \mathcal{H}^1(I^+(x,r))\right|, \left|\pi r - \mathcal{H}^1(I^-(x,r))\right|\right).$$

The Carleson ϵ^2-square function is given by

$$\mathcal{E}(x)^2 := \int_0^1 \epsilon(x,r)^2 \, \frac{dr}{r}. \tag{1.18}$$

CONJECTURE 1.9 (Carleson's ϵ^2-conjecture). *Suppose Γ is a Jordan curve. Except for a set of zero $\mathcal{H}^1$-measure, Γ has a tangent at $x \in \Gamma$ if and only if $\mathcal{E}(x) < \infty$.*

---

[8] The choice of p in the theorem will be made explicit in Chapter 3.

Note that if Γ is a line then $\mathcal{E}(x) = 0$ for all $x \in \Gamma$.

Thus the main result of Chapter 4, see Thereom 4.2, is a proof of this conjecture. The same result is also proven for a different type of domain, the so called 2-sided corkscrew domains.

Part 1

Sets with topology and the Analyst's TST

CHAPTER 2

Families of Analyst's TST

1. Introduction and statement of the results

This chapter will be devoted to give an answer to the question raised in Chapter 1, (3.1), namely if an estimate like (1.15), that is,

$$\operatorname{diam}(E)^d + \beta_E^{p,d}(E)^2 \sim \mathcal{H}^d(E) + \mathrm{BWGL(E)}.$$

could hold for quantitative properties different than being bilaterally approximated by a plane.

Before introducing the results of this chapter, let us state precisely the TST from [**AS18**]. It is phrased slightly differently from there, but we justify the reformulation in the appendix.

Theorem 2.1. *Let $1 \leq d < n$ and $E \subseteq \mathbb{R}^n$ be a closed set. Suppose that E is lower content d-regular with constant c_1 and let $\mathcal{D}$ denote the Christ-David*[1] *cubes for E. Let $\mathsf{C}_0 > 1$. Then there is $\epsilon > 0$ small enough so that the following holds. Let $1 \leq p < p(d)$ where*

$$p(d) := \begin{cases} \frac{2d}{d-2} & \text{if } d > 2 \\ \infty & \text{if } d \leq 2 \end{cases}. \tag{2.1}$$

For $R \in \mathcal{D}$, let

$$\mathrm{BWGL}(R) = \mathrm{BWGL}(R, \epsilon, \mathsf{C}_0) = \sum_{\substack{Q \in \mathrm{BWGL}(\epsilon, \mathsf{C}_0) \\ Q \subseteq R}} \ell(Q)^d.$$

and

$$\beta_{E,\mathsf{A},p}(R) := \ell(R)^d + \sum_{Q \subseteq R} \beta_E^{d,p}(\mathsf{A}\, B_Q)^2 \ell(Q)^d.$$

Then for $R \in \mathcal{D}$,

$$\mathcal{H}^d(R) + \mathrm{BWGL}(R, \epsilon, \mathsf{C}_0) \sim_{\mathsf{A},n,\mathsf{c}_1,p,\mathsf{C}_0,\epsilon} \beta_{E,\mathsf{A},p}(R). \tag{2.2}$$

Since all these values are comparable for all admissible values of A and p, below we will simply let

$$\beta_E(R) := \beta_{E,3,2}.$$

As mentioned in Chapter 1, the theorem above says that $\mathrm{BWGL}(R, \epsilon, \mathsf{C}_0)$ has some meaning if we compute the sum for a non-Ahlfors regular set: even though it does not necessarily satisfy a Carleson packing condition, it is comparable to the square sum of β's for any lower regular set. The next theorem, a consequence of our results (see Section 6 for its proof), says that also the sum $\mathrm{BAUP(R}, \epsilon, \mathsf{C}_0)$ keep a meaning outside the Ahlfors regular setting: the same one that $\mathrm{BWGL(R}, \epsilon, \mathsf{C}_0)$ has.

Theorem 2.2. *Let $E \subseteq \mathbb{R}^n$ be a lower content (d, c_1)-regular set with Christ-David cubes $\mathcal{D}$. For $R \in \mathcal{D}$, define*

$$\mathrm{BAUP}(R) := \mathrm{BAUP}(R, \mathsf{C}_0, \epsilon) = \sum_{\substack{Q \subseteq R \\ Q \in \mathrm{BAUP}(\mathsf{C}_0, \epsilon)}} \ell(Q)^d.$$

For all $R \in \mathcal{D}$, $\mathsf{C}_0 > 1$, and $\epsilon > 0$ small enough depending on C_0 and c,

$$\mathcal{H}^d(R) + \mathrm{BAUP}(R, \mathsf{C}_0, \epsilon) \sim_{\mathsf{C}_0,\epsilon,\mathsf{c}_1} \beta_E(R). \tag{2.3}$$

We mention one other geometric criteria studied by David and Semmes which we consider: the 'local symmetry' (LS) property is defined as follows. Given $\epsilon > 0$, let $\mathrm{LS}(R, \epsilon)$ be the sum of $\ell(Q)^d$ over those cubes in R for which there are[2] $y, z \in B_Q \cap E$ so that $\operatorname{dist}(2y - z, E) \geq \epsilon r$.

[1]For a definition of these, see Theorem 5.2 in the Appendix.

[2]For the notation B_Q, see below Theorem 5.2 in the Appendix.

THEOREM 2.3. *Let $E \subseteq \mathbb{R}^n$ be a lower content (d, c_1)-regular set and $\mathcal{D}$ its Christ-David cubes. Then for $\epsilon > 0$ small enough (depending on c), and $R \in \mathcal{D}$,*

$$\beta_E(R) \sim_{\mathsf{c}_1,\epsilon} \mathcal{H}^d(R) + \mathrm{LS}(R, \epsilon). \tag{2.4}$$

This may be surprising, since the Local Symmetry condition is dimensionless, that is, the integer d does not appear in the definition at all, and in fact it could be that, in the "good" cubes not featured in the sum, E could be very not flat and quite close in the Hausdorff distance to a $(d+1)$-dimensional cube, say, whereas the β-numbers measure the distance to a d-dimensional plane and would be large for these cubes. However, with the assumption that $\mathcal{H}^d(R)$ is finite, this prevents there being too many cubes where E is symmetric but looks like a $(d+1)$-dimensional surface (and this is natural considering that the proof in [DS91] connecting LS to flatness of the set uses the Ahlfors regularity of the sets they consider).

Our method for extending these results is quite flexible: the other characterizations of UR for which we prove analogous statements like those are the Local Convexity (LCV) and Generalized Weak Exterior Convexity (GWEC) conditions, although one could also consider other suitable characterizations in [DS93] as well. In fact, our main result is a general test for when a geometric criteria that guarantees uniform rectifiability (like BAUP or BWGL) also implies a result of the form Theorem 2.2. Its statement is a bit lengthy to give here, so we postpone it to Section 4. Loosely speaking, by a geometric criterion $\mathfrak{P}$, we mean a way of splitting up the surface cubes of a set E into "good" and "bad" cubes, the good cubes being those cubes near which E satisfies some condition that is trivially satisfied for a d-dimensional plane, like being close in the Hausdorff distance to a plane or union of planes. We say it guarantees UR if, whenever we have an Ahlfors regular set, a Carleson packing condition on the bad cubes implies UR. Our result, Lemma 2.25 below, states that if we have a geometric criterion that guarantees UR and it is, in some sense, continuous in the Hausdorff metric, then a result like Theorem 2.2 hold with BAUP replaced by $\mathfrak{P}$.

The main lemma that we use may be of independent interest, and has a few forthcoming applications to other problems (see for example [Azz] and Chapter 3) . For the reader familiar with uniform rectifiability, this result says that we can perform a Coronization of lower regular sets by Ahlfors regular sets in a way similar to how David and Semmes construct Coronizations of uniformly rectifiable sets by Lipschitz graphs (see [DS91, Chapter 2]).

MAIN LEMMA 2.4. *Let $k_0 > 0$, $\tau > 0$, $d > 0$ and E be a set that is lower content (d, c_1)-regular. Let $\mathcal{D}_k$ denote the Christ-David cubes on E of scale k and $\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k$. Let $Q_0 \in \mathcal{D}_0$ and $\mathcal{D}(k_0) = \bigcup_{k=0}^{k_0}\{Q \in \mathcal{D}_k | Q \subseteq Q_0\}$. Then we may partition $\mathcal{D}(k_0)$ into stopping-time regions $\mathsf{Tree}(R)$ for R from some collection $\mathsf{Top}(k_0) \subseteq \mathcal{D}(k_0)$ with the following properties:*

(1) We have

$$\sum_{R\in\mathsf{Top}(k_0)} \ell(R)^d \lesssim_{\mathsf{c}_1,d} \mathcal{H}^d(Q_0). \tag{2.5}$$

(2) Given $R \in \mathsf{Top}(k_0)$ and a stopping-time region $\mathsf{Str} \subseteq \mathsf{Tree}(R)$ with maximal cube T, let $\mathcal{F}$ denote the minimal cubes of Str and

$$d_{\mathcal{F}}(x) = \inf_{Q\in\mathcal{F}} (\ell(Q) + \mathrm{dist}(x, Q)) \tag{2.6}$$

For $C_0 > 4$ and $\tau > 0$, there is a collection $\mathscr{C}$ of disjoint dyadic cubes covering $C_0 B_T \cap E$ so that if

$$E(\mathsf{Str}) = \bigcup_{I\in\mathscr{C}} \partial_d I,$$

where $\partial_d I$ denotes the d-dimensional skeleton of I, then the following hold:

(a) $E(\mathsf{Str})$ is Ahlfors regular with constants depending on $C_0, \tau, d,$ and c_1.

(b) We have the containment

$$C_0 B_T \cap E \subseteq \bigcup_{I\in\mathscr{C}} I \subseteq 2C_0 B_T. \tag{2.7}$$

(c) E is close to $E(\mathsf{Str})$ in $C_0 B_T$ in the sense that

$$\mathrm{dist}(x, E(\mathsf{Str})) \lesssim \tau d_{\mathcal{F}}(x) \quad \textit{for all } x \in E \cap C_0 B_T. \tag{2.8}$$

(*d*) *The cubes in* $\mathscr{C}$ *satisfy*

$$\ell(I) \sim \tau \inf_{x \in I} d_{\mathcal{F}}(x) \text{ for all } I \in \mathscr{C}. \tag{2.9}$$

The last inequality says that the cubes in $\mathscr{C}$ are distributed in a sort of Whitney fashion. In particular, if two cubes in $\mathscr{C}$ are adjacent, then they have comparable sizes.

Observe that the constants don't depend on k_0. The presence of k_0 is an artifact of the proof, but in applications we will take $k_0 \to \infty$.

1.1. Outline. In Section 3, we prove the Main Lemma and show that a general lower regular set can be approximated by Alhfors regular sets. In Section 4, we show how, if the sum of cubes where a geometric criteria like the BAUP is finite, then we can actually make these Ahlfors regular sets uniformly rectifiable. Using a result of David and Semmes, we know that the sum of β's will be finite for these sets, and then that will imply the β's for the original set are summable by approximation. After that, we apply our works to get results similar to the Traveling Saleman, but with BWGL replaced by other geometric criteria. In Section 5, we show the same result holds with BWGL replaced by the Local Symmetry and Local Convexity conditions. In Section 6, we consider the BAUP condition and prove Theorem 2.2, and in Section 7, we study the GWEC.

1.2. Acknowledgements. This chapter is based on the paper [**AV19**], which was coauthored with Jonas Azzam. We'd like to thank Raanan Schul for his useful input during the writing of [**AV19**].

2. Preliminaries

In the upcoming section, we will use both dyadic cubes and Christ-David cubes (as in the appendix, Theorem 5.2). Let us set some notation. We will denote real dyadic cube by I, J and Christ-David cubes by Q, R, P. For more notational convention on Christ-David cubes, see the Appendix.

As for dyadic cubes, we will denote the family of dyadic cubes in $\mathbb{R}^n$ by Δ; for $k \in \mathbb{Z}$, Δ_k will denote the subfamily of those dyadic cubes with sidelength 2^{-k}. Given a dyadic cube I_0, we will write $\Delta(I_0)$ to denote the subfamily of dyadic cubes which are contained in I_0. Given some $m \in \mathbb{Z}$, we will write

$$\Delta^m := \bigcup_{k=m}^{\infty} \Delta_k,$$

that is, Δ^m is the family of dyadic cubes with side length at least 2^m. We will also write

$$\Delta^m(I_0) := \Delta^m \cap \Delta(I_0).$$

Finally, given a dyadic cube I, we denote by $n(I)$ the integer number so that

$$\ell(I) = 2^{n(I)}. \tag{2.10}$$

For a cube $I \in \Delta$, we write

$$\partial_d I \tag{2.11}$$

to denote *the d-dimensional skeleton* of I. Given a dyadic cube I in $\mathbb{R}^n$, the d-dimensional skeleton of I is just the union of all its d-dimensional faces. Let us remark that for a set V, we write ∂V to mean the standard boundary of V; so in particular $\partial I = \partial_{n-1} I$.

Remark 2.5. We may also use the notation Δ_m to mean the family of cubes with $\ell(I) = 2^{-m}$.

3. Proof of the Main Lemma

Let E and Q_0 be as in the Main Lemma. Notice that Q_0 is also a lower regular set, although it may not be closed, but we will not need that. We split the proof into a few subsections.

3.1. Frostmann's Lemma. The first step of the proof follows the proof Frostmann's lemma[3] , but with some extra care. Let $I_0 := [0,1]^n$. Without loss of generality, we can assume $Q_0 \subseteq I_0$ and $\operatorname{diam} Q_0 \geq \ell(I_0)$.

For $k \in \mathbb{Z}$, let

$$\Delta_k(Q_0) = \{I \in \Delta_k \mid I \cap Q_0 \neq \emptyset\}, \quad \Delta^k(Q_0) = \bigcup_{j=0}^{k} \Delta_j(Q_0), \quad \Delta(Q_0) = \Delta^\infty(Q_0)$$

and

$$E_k = \bigcup_{I \in \Delta_k(Q_0)} I.$$

Let $m \in \mathbb{N}$[4] (we will choose it later). First let $\mu_m^m = \mathscr{H}^n|_{E_m} 2^{(n-d)m}$. In this way,

$$\mu_m^m(I) = \ell(I)^d \quad \text{for all } I \in \Delta_m(Q_0). \tag{2.12}$$

We define a set of cubes Bad (which depends on m) as follows. First, we immediately add $\Delta_m(Q_0)$ to Bad. Next, for each $I \in \Delta_{m-1}(Q_0)$, if

$$\mu_m^m(I) > 2\ell(I)^d,$$

then we add I to Bad and define

$$\mu_m^{m-1}|_I = \ell(I)^d \frac{\mu_m^m|_I}{\mu_m^m(I)} < \frac{1}{2}\mu_m^m|_I$$

Otherwise, we set

$$\mu_m^{m-1}|_I = \mu_m^m|_I.$$

Inductively, suppose we have defined μ_m^{k+1} for some integer $k < m$. For $I \in \Delta_k(E)$, if

$$\mu_m^{k+1}(I) > 2\ell(I)^d,$$

then place $I \in \mathsf{Bad}$ and set

$$\mu_m^k|_I = \ell(I)^d \frac{\mu_m^{k+1}|_I}{\mu_m^{k+1}(I)} < \frac{1}{2}\mu_m^{k+1}|_I. \tag{2.13}$$

Otherwise, we set

$$\mu_m^k|_I = \mu_m^{k+1}|_I. \tag{2.14}$$

Finally, we put $I_0 \in \mathsf{Bad}$.

Given a cube $I \in \Delta$, recall that $n(I)$ is the integer such that $\ell(I) = 2^{n(I)}$. Moreover, for $J \in \Delta^m(Q_0)$ (so that $\ell(J) \geq 2^{-m}$), we let $b(J)$ to be the number of cubes from Bad containing J; if J is itself a bad cube, $b(J)$ will account for this. Now, again with $J \in \Delta^m(Q_0)$, let $I_0, ..., I_{b(J)} \in \mathsf{Bad}$ be all the bad cubes containing J so that $I_j \supseteq I_{j+1}$. With this notation, we see that $b(I_j) = j$ for all j, and if a dyadic cube $J \in \mathsf{Bad}$, then $J = J_{b(J)}$. Note in particular that this is consistent with I_0 being in Bad.

Let now $I \in \mathsf{Bad}$ and $J \in \Delta_m(Q_0)$ (and this $\ell(J) = 2^{-m}$) so that $J \subset I$. We write[5]

$$\begin{aligned}\mu_m^{n(I)}(J) = \mu_m^{n(I_{b(I)})}(J) &\overset{(2.13)}{<} \frac{1}{2}\mu_m^{n(I_{b(I)})+1}(J) \overset{(2.14)}{=} \frac{1}{2}\mu_m^{n(I_{b(I)+1})}(J) < \cdots \\ \cdots &< \frac{1}{2^{b(J)-b(I)}}\mu_m^{n(I_{b(J)})}(J) = \frac{1}{2^{b(J)-b(I)}}\mu_m^m(J) \overset{(2.12)}{=} \frac{\ell(J)^d}{2^{b(J)-b(I)}}.\end{aligned} \tag{2.15}$$

[3] Let us recall what the Frostman's Lemma and its converse say: Let B a Borel set in $\mathbb{R}^n$. Then $\mathcal{H}^s(B) > 0$ if and only if there exists a finite Radon measure μ such that $\mu(B(x,r)) \leq r^s$ for $x \in \mathbb{R}^n$ and $r > 0$. Moreover, we can find a μ such that $\mu(B) \geq c\mathcal{H}_\infty(B)$, where c depends only on n. See [**Mat95**], Theorem 8.8.

[4] In the following Remark 2.5 will apply.

[5] Let us give a quick explanation of the calculations below.

(1) The first equality holds because I is in Bad and thus we have $I = I_{b(I)}$.
(2) The next equality and inequality are clear.
(3) The fourth inequality is by reiteration.
(4) The next one follows from $I_{n(J)} = J$.

Finally, observe that since Q_0 is lower content (d, c_1)-regular, if $J \cap Q_0 \neq \emptyset$ and $J \in \Delta_m(Q_0)$, then

$$\ell(J)^d \lesssim_{\mathsf{c}_1} \mathcal{H}^d_\infty(3J \cap Q_0) \le \mathcal{H}^d(3J \cap Q_0). \tag{2.16}$$

With this, we can show the first conclusion of Main Lemma 2.4, i.e. (2.5).

$$\begin{aligned}
\sum_{I \in \mathsf{Bad}} \ell(I)^d &\overset{(2.13)}{=} \sum_{I \in \mathsf{Bad}} \mu_m^{n(I)}(I) = \sum_{I \in \mathsf{Bad}} \sum_{\substack{J \in \Delta_m(Q_0) \\ J \subseteq I}} \mu_m^{n(I)}(J) \\
&\overset{(2.15)}{<} \sum_{I \in \mathsf{Bad}} \sum_{\substack{J \in \Delta_m(Q_0) \\ J \subseteq I}} \frac{\ell(J)^d}{2^{b(J)-b(I)}} \\
&= \sum_{J \in \Delta_m(Q_0)} \ell(J)^d \sum_{\substack{I \in \mathsf{Bad} \\ I \supseteq J}} 2^{-b(J)+b(I)} \lesssim \sum_{J \in \Delta_m(Q_0)} \ell(J)^d \\
&\overset{(2.16)}{\lesssim}_{\mathsf{c}_1} \sum_{J \in \Delta_m(Q_0)} \mathcal{H}^d(3J \cap Q_0)^d \lesssim \mathcal{H}^d(Q_0).
\end{aligned} \tag{2.17}$$

The last inequality follows since the cubes $3J$ have bounded overlap.

For $I \in \mathsf{Bad}$, let

$$\mu^I := \mu_m^{n(I)}|_I.$$

Note that by construction[6], for each $J \subseteq I$, we have that

$$\mu^I(J) \le 2\ell(J)^d, \tag{2.18}$$

and thus this also holds for all dyadic cubes J, even when $J \supseteq I$ or $J \cap I = \emptyset$. In particular, since any ball $B(x, r)$ can be covered by boundedly many dyadic cubes of size comparable to r, we obtain that

$$\mu^I(B(x, r)) \lesssim r^d \text{ for all } x \in \mathbb{R}^n,\ r > 0. \tag{2.19}$$

Moreover,

$$\mu^I(I) = \ell(I)^d.$$

Remark 2.6. Ideally what we would want to do at this stage is to find, for each $I \in \mathsf{Bad}$, the maximal bad cubes $I_j \in \mathsf{Bad}$ properly contained in I and define a set like

$$E_I = \bigcup_j \partial_d I_j$$

where $\partial_d J$ is the d-dimensional skeleton of a cube J. Then one can use μ^I to show that E_I is an Ahlfors regular set. This would give an Ahlfors regular set which approximate Q_0 at the scale of I. However, the collection E_I will not be suitable for the applications we have in mind, since we need that the sizes of the cubes whose skeletons form E_I don't vary too wildly (that is, adjacent cubes should have comparable sizes). This is why more work is needed.

3.2. Trees. For $I \in \mathsf{Bad}$, we will let $\mathsf{Tree}(I)$ to be those cubes in Δ contained in I for which the smallest cube from Bad that they are properly contained in is I, and we will let $\mathsf{Stop}(I)$ be those cubes from Bad in $\mathsf{Tree}(I)$ properly contained in I.

Remark 2.7. Observe that $\mathsf{Stop}(I) \subseteq \mathsf{Tree}(I)$, and while the collections $\{\mathsf{Tree}(I) : I \in \mathsf{Bad}\}$ do not form a disjoint partition of $\Delta^m(Q_0)$, they do cover $\Delta^m(Q_0)$, and they only intersect at the top cubes and stopped cubes.

Lemma 2.8. *For $I \in \mathsf{Bad}$ and $J \in \mathsf{Stop}(I)$,*

$$2^{d-n-1}\ell(J)^d \le \mu^I(J) \le 2\ell(J)^d \tag{2.20}$$

[6]This can be observed by starting with $\mu_m^{b(J)} = \ell(J)^d$ and following through the construction. Indeed, this upper bound is what the proof of the Frostmann's lemma tries to achieve.

Proof. The second indequality is just (2.18). One the other hand, for $I \in \mathsf{Bad}$ we have[7]

$$\mu_m^{n(I)+1}(I) = \sum_{\substack{J\in\Delta_{n(I)+1}\\ J\subseteq I}} \mu_m^{n(I)+1}(J) \leq \sum_{\substack{J\in\Delta_{n(I)+1}\\ J\subseteq I}} 2\ell(J)^d$$
$$= \sum_{\substack{J\in\Delta_{n(I)+1}\\ J\subseteq I}} 2^{1-d}\ell(I)^d \leq 2^{n-d+1}\ell(I)^d$$

and for $J \in \mathsf{Stop}(I)$,

$$\mu_m^{n(I)+1}(J) = \mu_m^{n(J)}(J) = \ell(J)^d.$$

Thus,

$$2\ell(J)^d \overset{(2.18)}{\geq} \mu^I(J) = \mu_m^{n(I)}(J) \overset{(2.13)}{=} \mu_m^{n(I)+1}(J)\frac{\ell(I)^d}{\mu_m^{n(I)+1}(I)} \geq 2^{d-n-1}\ell(J)^d.$$

□

Let $M > 1$, to be chosen soon. For $Q \in \mathcal{D}(k_0)$ (as defined in the statement of main lemma 2.4) and $I \in \Delta(Q_0)$, we write $Q \sim I$ if

(2.21) $$MB_Q \cap I \neq \emptyset \text{ and } \lambda\ell(I) \leq \ell(Q) < \ell(I)$$

where λ is as in Theorem 5.2. Observe that for m large enough,

(2.22) $$\{I \in \Delta(Q_0) : I \sim Q \text{ for some } Q \in \mathcal{D}(k_0)\} \subseteq \Delta^m(Q_0).$$

Indeed, If $Q \in \mathcal{D}(k_0)$, this means $\ell(I) \geq \lambda\ell(Q) \geq 5\lambda^{k_0+1} > 2^{-m}$ for m large enough, and now we just recall Remark 2.7.

We now perform the following stopping-time algorithm on the cubes $\mathcal{D}(k_0)$. For $R \in \mathcal{D}(k_0)$ contained in Q_0, we let $\mathsf{Stop}(R)$ denote the set of maximal cubes in R from $\mathcal{D}(k_0)$ that are either in $\mathcal{D}_{k_0}$ or have a child Q for which there is $I \in \mathsf{Bad}$ such that $Q \sim I$. Observe that if $R \in \mathcal{D}_{k_0}$, then $\mathsf{Stop}(R) = \{R\}$. We then let $\mathsf{Tree}(R)$ be those cubes contained in R that are not properly contained in any cube from $\mathsf{Stop}(R)$, so in particular, $\mathsf{Stop}(R) \subseteq \mathsf{Tree}(R)$. Let $\mathsf{Next}(R)$ be the children of cubes in $\mathsf{Stop}(R)$ that are also in $\mathcal{D}(k_0)$ (so this could be empty).

Now let $\mathsf{Top}_0 = \{Q_0\}$, and for $R \in \mathsf{Top}_k$, we let

$$\mathsf{Top}_{k+1} = \bigcup_{R\in\mathsf{Top}_k} \mathsf{Next}(R),$$

that is, Top_{k+1} are the children of the cubes in $\mathsf{Stop}(R)$ for each $R \in \mathsf{Top}_k$. Let

$$\mathsf{Top} = \bigcup_{k\geq 0} \mathsf{Top}_k.$$

Note that for each $R \in \mathsf{Top}$, if R^1 is its parent, then $R^1 \in \mathsf{Stop}(R')$ for some cube R', and so there is $I_R \in \mathsf{Bad}$ with $I_R \sim R''$ for some sibling $R'' \in \mathsf{Child}(R^1)$. In particular, the map $R \mapsto I_R$ maps boundedly many cubes to one cube, and so

(2.23) $$\sum_{R\in\mathsf{Top}} \ell(R)^d \lesssim_M \sum_{I\in\mathsf{Bad}} \ell(I)^d \overset{(2.17)}{\lesssim}_N \mathcal{H}^d(Q_0).$$

The collection Top is our desired collection and $\{\mathsf{Tree}(R) \mid R \in \mathsf{Top}\}$ are the desired stopping-time regions for the Main Lemma and (2.5) now follows from (2.23). It remains to verify items (2) of the Main Lemma, which will be the focus of the next two subsections. We will first need a lemma about our trees.

Lemma 2.9. *Let $R \in \mathsf{Top}$ and*

$$S(R) := \{I \in \Delta(Q_0) |\ Q \sim I \textit{ for some } Q \in \mathsf{Tree}(R)\}.$$

Then there is $N_0 \lesssim_{n,M}$ and $J_1(R), ..., J_{N_0}(R) \in \mathsf{Bad}$ so that

$$S(R) \subseteq \mathsf{Tree}(J_1(R)) \cup \cdots \cup \mathsf{Tree}(J_{N_0}(R)).$$

[7]We also use the fact that there are 2^n-dyadic cubes $J \subseteq I$ with $\ell(J) = \ell(I)/2$

PROOF. Consider the cubes $I_1, ..., I_{N_0}$ in $\Delta(Q_0)$ of maximal size so that $I_j \sim R$ (note that N_0 here depends only on n and M). Then for m large enough, each I_j is contained in $\Delta^m(Q_0)$, by (2.22), and therefore in $\mathsf{Tree}(J_j)$ for some $J_j \in \mathsf{Bad}$, by Remark 2.7.

Now let $I \in S(R)$; by definition there is $Q \in \mathsf{Tree}(R)$ satisfying (2.21). Then $I \subseteq I_j \subseteq J_j$ for some j. If $I \notin \mathsf{Tree}(J_j)$, then there is $J \in \mathsf{Stop}(J_j)$ so that $I \subsetneq J \subseteq I_j \subseteq J_j$. Since $I_j \sim R$, $\ell(I_j) < \ell(R)$, so we must have $\ell(J) < \ell(R)$. Thus, if Q' is the maximal ancestor of Q with $\ell(Q') < \ell(J)$, then $\ell(Q') < \ell(R)$, and so $Q' \subsetneq R$. Since $Q \in \mathsf{Tree}(R)$, this implies $Q' \in \mathsf{Tree}(R)$. Since $Q \sim I$ and $\lambda \ell(J) \leq \ell(Q') < \ell(J)$ by the maximality of Q', we also have $Q' \sim J$. So the parent $Q'' \subseteq R$ of Q' must be in $\mathsf{Stop}(R)$, but this contradicts Q being in $\mathsf{Tree}(R)$. We let $J_j(R) = J_j$ and this proves the lemma. $\square$

3.3. Smoothing. We follow the "smoothing" process of David and Semmes (c.f. [**DS91**, Chapter 8]). Fix $0 < \tau < 1$. For a finite family of cubes $\mathcal{F} \subset \mathcal{D}$, define the following *smoothing* function: for a point $x \in \mathbb{R}^n$, set

$$d_{\mathcal{F}}(x) := \inf_{S \in \mathcal{F}} \left(\ell(S) + \operatorname{dist}(x, S)\right), \tag{2.24}$$

and for a dyadic cube $I \in \Delta$,

$$d_{\mathcal{F}}(I) := \inf_{x \in I} d_{\mathcal{F}}(x) = \inf_{S \in \mathcal{F}} \left(\ell(S) + \operatorname{dist}(I, S)\right). \tag{2.25}$$

We define $\mathscr{C}_{\mathcal{F}}$ to be the set of maximal cubes $I \in \Delta(Q_0)$ for which

$$\ell(I) < \tau d_{\mathcal{F}}(I). \tag{2.26}$$

The following lemmas are quite standard and appear in different forms depending on the scenario in which they are being applied. For example, we will carry out a similar smoothing process later on in Chapter 4; see also [**DS91**, Lemma 8.7] for a further example. We include their proofs below for completeness.

LEMMA 2.10. *Let $I, I' \in \Delta$. Then,*

$$d_{\mathcal{F}}(I) \leq 2\ell(I) + \operatorname{dist}(I, I') + 2\ell(I') + d_{\mathcal{F}}(I'). \tag{2.27}$$

PROOF. Let $x, y \in I$ and $x', y' \in I'$. Let also $Q \in \mathcal{F}$; we have

$$d_{\mathcal{F}}(x) \leq |x - y| + |y - y'| + |y' - x'| + \operatorname{dist}(x', Q) + \ell(Q), \tag{2.28}$$

simply by triangle inequality and the definition of $d_{\mathcal{F}}$. Clearly, $|y - y'| \leq \operatorname{dist}(I, I')$; moreover, infimising first over all $Q \in \mathcal{F}$ and then over all $x' \in I$, we obtain (2.27). $\square$

LEMMA 2.11. *Let $I \in \mathscr{C}_{\mathcal{F}}$; then*

$$\frac{\tau}{2} d_{\mathcal{F}}(I) \leq \ell(I) < \tau d_{\mathcal{F}}(I). \tag{2.29}$$

PROOF. By (2.26), $\ell(I) < \tau d_{\mathcal{F}}(I)$, and by definition it is a maximal cube satisfying this inequality. Hence if $\widehat{I}$ is the parent of I, we see that

$$2\ell(I) = \ell(\widehat{I}) \geq \tau d_{\mathcal{F}}(\widehat{I}) \geq \tau d_{\mathcal{F}}(I).$$

$\square$

The following lemma says that if two cubes in $\mathscr{C}_{\mathcal{F}}$ are close to each other, then they have comparable size.

LEMMA 2.12. *Let $I, J \in \mathscr{C}_{\mathcal{F}}$ and recall that $\mathscr{C}_{\mathcal{F}}$ depends on a parameter τ. Let $0 < \eta < 1$ be another small parameter. If*

$$\eta^{-1} J \cap \eta^{-1} I \neq \emptyset, \tag{2.30}$$

for $\tau^{-1} > 2\sqrt{n}/\eta$,

$$\ell(I) \sim \ell(J). \tag{2.31}$$

PROOF. It suffices to show that for all $y \in \eta^{-1} J$,

$$\tau^{-1} \ell(J) \sim d_{\mathcal{F}}(y) \tag{2.32}$$

Since $d_{\mathcal{F}}$ is 1-Lipschitz, we see that

$$|d_{\mathcal{F}}(J) - d_{\mathcal{F}}(y)| \leq \eta^{-1} \operatorname{diam}(J) = \frac{\sqrt{n}}{\eta} \ell(J).$$

Hence if $\tau^{-1} > 2\sqrt{n}/\eta$,

$$d_{\mathcal{F}}(y) \geq d_{\mathcal{F}}(J) - \frac{\sqrt{n}}{\eta}\ell(J) \geq \left(\tau^{-1} - \frac{\sqrt{n}}{\eta}\right)\ell(J) \geq \frac{1}{2\tau}\ell(J) \tag{2.33}$$

On the other hand, again using the fact that $d_{\mathcal{F}}$ is 1-Lipschitz, we see that

$$d_{\mathcal{F}}(y) \lesssim (\eta^{-1} + \tau^{-1})\ell(J) \lesssim \tau^{-1}\ell(J). \tag{2.34}$$

$\square$

3.4. Constructing an Ahlfors regular set with respect to a tree. Let $R \in \mathsf{Top}$ and $\mathsf{Str} \subseteq \mathsf{Tree}(R)$ be a stopping-time region, let T denote the maximal cube in Str, $\mathcal{F}$ be the set of minimal cubes of Str (that is, those cubes in Str that don't properly contain another cube in Str).

Observe that since all the cubes we are working with come from $\mathcal{D}(k_0)$ and the number of these cubes in Q_0 is finite, the infimum $d_{\mathcal{F}}$ is attained, and so for each $I \in \Delta$ there is $Q_I \in \mathcal{F}$ so that

$$d_{\mathcal{F}}(I) = \ell(Q_I) + \operatorname{dist}(Q_I, I). \tag{2.35}$$

Let $C_0 > 4$ and set

$$\widehat{T} = \bigcup\{Q \in \mathcal{D} \mid \ell(Q) = \ell(T),\ Q \cap C_0 B_T \neq \emptyset\}, \tag{2.36}$$

$$\mathscr{C} = \{I \in \mathscr{C}_{\mathcal{F}} \mid I \cap \widehat{T} \neq \emptyset\}, \tag{2.37}$$

and

$$\widehat{E} := \bigcup_{I \in \mathscr{C}} \partial_d I. \tag{2.38}$$

The set $\widehat{E}$ will be our desired $E(\mathsf{Str})$ as in the statement of the Main Lemma (we just write $\widehat{E}$ for short).

LEMMA 2.13. *For $m \in \mathbb{N}$ large enough,*

$$\mathscr{C} \subseteq \Delta^m. \tag{2.39}$$

PROOF. Note that by (2.29), and because $Q_I \in \mathcal{D}(k_0)$, for $I \in \mathscr{C}$,

$$\ell(I) \geq \frac{\tau}{2} d_{\mathcal{F}}(I) \geq \frac{\tau}{2}\ell(Q_I) \geq \frac{5\tau}{2}\lambda^{k_0},$$

and for τ small enough,

$$\ell(I) < \tau d_{\mathcal{F}}(I) \leq \tau(\ell(T) + \operatorname{dist}(I,T)) \leq \tau(C_0+1)\ell(T) < \frac{1}{5}\ell(Q_0) \leq 1.$$

Thus, (2.39) follows for m large enough from these two inequalities. $\square$

REMARK 2.14. Note that we definitely don't have that $\mathscr{C} \subseteq \Delta^m(Q_0)$, since some cubes in $\mathscr{C}$ are actually disjoint from Q_0. This will cause some difficulties later.

LEMMA 2.15. *Part 2.(b) of the Main Lemma holds, that is, we have the containment*

$$C_0 B_T \cap E \subset \bigcup_{I \in \mathscr{C}} I \subset 2C_0 B_T.$$

PROOF. Firstly, as $C_0 B_T \cap E \subseteq \widehat{T}$, we immediately have the first containment, so we just need to show the second containment.

Note that if $I \in \mathscr{C}$, then $I \cap Q \neq \emptyset$ for some $Q \in \mathcal{D}$ with $\ell(Q) = \ell(T)$ and $Q \cap C_0 B_T \neq \emptyset$. Thus,

$$\operatorname{dist}(I,T) \leq \operatorname{dist}(Q,T) + \operatorname{diam} Q \leq C_0\ell(T) + 2\ell(T) < (C_0+2)\ell(T).$$

Thus,

$$\operatorname{diam} I = \sqrt{n}\ell(I) < \tau d_{\mathcal{F}}(I) \leq \tau(\operatorname{dist}(I,T) + \ell(T)) < \tau(C_0+3)\ell(T)$$

so for $\tau > 0$ small, $\operatorname{diam} I \leq \frac{C_0}{2}\ell(T)$. Thus, $I \subseteq (3C_0/2+2)B_T \subseteq 2C_0 B_T$, which proves the lemma. $\square$

LEMMA 2.16. *Part 2.(c) of the Main Lemma holds, that is, for all $x \in E \cap C_0 B_T$ we have*

$$\operatorname{dist}(x, \widehat{E}) \lesssim \tau d_{\mathcal{F}}(x).$$

PROOF. Let $x \in E \cap C_0 B_T \subseteq \widehat{T}$. By part (b), there is I so that $x \in I \in \mathscr{C} \subseteq \mathscr{C}_{\mathcal{F}}$. By definition, $\partial_d I \subseteq \widehat{E}$, and so

$$\operatorname{dist}(x, \widehat{E}) \leq \operatorname{diam}(I) \leq \sqrt{n}\ell(I) < \sqrt{n}\tau d_{\mathcal{F}}(I) \leq \sqrt{n}\tau d_{\mathcal{F}}(x).$$

□

Moreover, (2.9) follows from (2.29). Thus, to prove the Main Lemma, all that remains to be shown is the following lemma.

LEMMA 2.17. *Part 2.(a) of the Main Lemma holds, that is, the set $\widehat{E}$ is Ahlfors d-regular.*

PROOF. Let $x \in \widehat{E}$ and $0 < r < \operatorname{diam} \widehat{E} \leq 2C_0\ell(T)$. We define

$$\mathscr{C}(x,r) = \{I \in \mathscr{C} \,|\, I \cap B(x,r) \neq \emptyset\}.$$

We split into three cases, and in each case we prove first the upper estimate for being Ahlfors regular (i.e. $\mathcal{H}^d(\widehat{E} \cap B(x,r)) \lesssim r^d$) and then the lower estimate.

Case 1: $2r \leq d_{\mathcal{F}}(x)$. Since $d_{\mathcal{F}}$ is Lipschitz, this means $d_{\mathcal{F}}(y) \geq d_{\mathcal{F}}(x) - |x-y|$, and so if $I \in \mathscr{C}(x,r)$, $y \in I$ is so that $d_{\mathcal{F}}(I) = d_{\mathcal{F}}(y)$, and $z \in I \cap B(x,r)$, then $|z-y| \leq \operatorname{diam} I = \sqrt{n}\ell(I)$, and hence

$$\begin{aligned} \ell(I) &\overset{(2.29)}{\sim} \tau d_{\mathcal{F}}(I) = \tau d_{\mathcal{F}}(y) \geq \tau(d_{\mathcal{F}}(x) - |x-y|) \\ &\geq \tau(2r - |x-z| - |z-y|) \gtrsim \tau(r - \sqrt{n}\ell(I)). \end{aligned}$$

Thus for $\tau \ll \sqrt{n}$ we have $\ell(I) \gtrsim \tau r$. This implies $\#\mathscr{C}(x,r) \lesssim_{n,\tau} 1$, and so it is not hard to show that

$$\mathscr{H}^d(\widehat{E} \cap B(x,r)) \sim_{n,\tau} 1.$$

Case 2: $8\ell(T) > 2r > d_{\mathcal{F}}(x)$.

Before we proceed, we record a few estimates. First, for $I \in \mathscr{C}(x,r)$, if $2r > d_{\mathcal{F}}(x)$,

$$\tau^{-1}\ell(I) \overset{(2.29)}{<} d_{\mathcal{F}}(I) \leq d_{\mathcal{F}}(y) \leq d_{\mathcal{F}}(x) + |x-y| < 2r + r = 3r \tag{2.40}$$

Next, note that for all $I \in \mathscr{C}$, $\ell(Q_I) \leq d_{\mathcal{F}}(I)$. Let Q_I' be the largest cube in Str containing Q_I so that $\ell(Q_I') \leq d_{\mathcal{F}}(I)$.

LEMMA 2.18. *If $x \in \widehat{E}$ and $d_{\mathcal{F}}(x) < 2r < 24\ell(T)$, then*

$$\ell(Q_I') \sim_\tau \ell(I). \tag{2.41}$$

PROOF. If $Q_I' = T$, then

$$\ell(T) = \ell(Q_I') \leq d_{\mathcal{F}}(I) \overset{(2.40)}{<} 3r \lesssim \ell(T)$$

and so

$$\ell(Q_I') = \ell(T) \sim d_{\mathcal{F}}(I) \sim \tau^{-1}\ell(I) \tag{2.42}$$

Otherwise, if $\ell(Q_I') < \ell(T)$, then $\ell(Q_I') \sim d_{\mathcal{F}}(I) \overset{(2.29)}{\sim}_\tau \ell(I)$ by maximality of Q_I' (indeed, if $\ell(Q_I') < \lambda d_{\mathcal{F}}(I)$, then its parent Q_I'' satisfies $\ell(Q_I'') < d_{\mathcal{F}}(I)$ and $Q_I'' \in \mathsf{Str}$ since $Q_I' \subsetneq T$, but this contradicts the maximality of Q_I'). This proves the lemma. □

Recall (2.39) and let (See Figure 1)

$$\begin{aligned} \mathscr{C}_1(x,r) &= \{I \in \mathscr{C}(x,r) : I \cap Q_0 \neq \emptyset\} = \mathscr{C}(x,r) \cap \Delta^m(Q_0), \\ \mathscr{C}_2(x,r) &= \mathscr{C}(x,r) \backslash \mathscr{C}_1(x,r). \end{aligned}$$

LEMMA 2.19. *If $x \in \widehat{E}$ and $d_{\mathcal{F}}(x) < 2r < 24\ell(T)$, then*

$$\sum_{I \in \mathscr{C}_1(x,r)} \ell(I)^d \lesssim r^d. \tag{2.43}$$

PROOF. We need an estimate like $\ell(I)^d \lesssim \mathcal{H}^d_\infty(I \cap Q_0)$, but this may not necessarily be true: of course $I \cap Q_0 \neq \emptyset$ since $I \in \mathscr{C}_1(x,r)$, but it could be that I only intersects Q_0 at a corner of I so $\mathcal{H}^d_\infty(I \cap Q_0)$ could be very small compared to $\ell(I)^d$. To overcome this, we associate to I a neighboring dyadic cube that does intersect E in a large set. Let Nei(I) be the

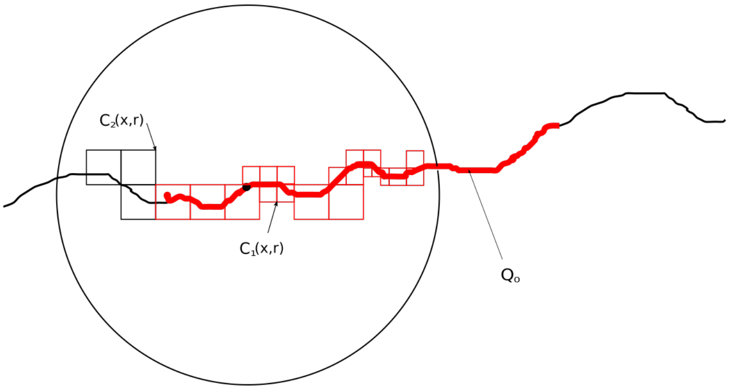

FIGURE 1. The set of cubes $\mathscr{C}_1(x,r)$ and $\mathscr{C}_2(x,r)$.

set of dyadic cubes $J \subseteq 3I$ with $\ell(J) = \ell(I)$. Then

$$\ell(I)^d \lesssim \mathcal{H}^d_\infty(3I \cap Q_0) \leq \sum_{J \in \mathrm{Nei(I)}} \mathcal{H}^d_\infty(J \cap Q_0).$$

Hence there is $I' \in$ Nei(I) so that

$$\mathcal{H}^d_\infty(I' \cap Q_0) \gtrsim \ell(I)^d.$$

We claim that $I' \sim Q'_I$ (as defined in (2.21)). Indeed, since $I' \subseteq 3I$, we know that

$$\mathrm{dist}(I', Q'_I) \leq \mathrm{diam}\, I + \mathrm{dist}(I, Q_I) \leq \sqrt{n}\ell(I) + d_{\mathcal{F}}(I) \overset{(2.29)}{\lesssim} \tau^{-1}\ell(I) \overset{(2.42)}{\sim} \ell(Q'_I).$$

Moreover, we also have that $\ell(I) \sim_\tau \ell(Q'_I)$; thus, for a sufficiently large choice of M (in particular $M \gg \tau^{-1}$), we obtain $MB_{Q'_I} \cap I' \neq \emptyset$. Moreover, since $Q'_I \in \mathsf{Str} \subset \mathsf{Tree}(R)$ and $I' \in \Delta^m(Q_0)$ (because $I' \cap Q_0 \neq \emptyset$ and $I \in \Delta^m$ by (2.39)), this implies $I' \sim Q'_I$, and so $I' \in S(R)$ (where $S(R)$ is as in Lemma 2.9). In particular, there is $J_i = J_i(R)$ so that $I' \in \mathsf{Tree}(J_i)$ by Lemma 2.9. We will use this fact shortly, but we need one more estimate: we now claim that

$$\sum_{I \in \mathscr{C}_1(x,r)} \mathbb{1}_{I'} \lesssim \mathbb{1}_{B(x,2r)}. \tag{2.44}$$

Indeed, if $y \in I'_1 \cap \cdots \cap I'_\ell$ for some distinct $I_1, ..., I_\ell \in \mathscr{C}_1(x,r)$, then the I_j are disjoint and $y \in 3I_1 \cap \cdots \cap 3I_\ell$, so Lemma 2.12 implies they have sizes all comparable to I_1 and are also contained in $9I_1$ (assuming I_1 is the largest). Thus if $|A|$ denotes the Lebesgue measure of a set A,

$$\ell|I_1| \sim_n \sum_{i=1}^\ell |I_i| = \left|\bigcup_{i=1}^\ell I_i\right| \leq |9I_1|$$

which implies $\ell \lesssim 1$, thus, $\sum_{I \in \mathscr{C}_1(x,r)} \mathbb{1}_{I'} \lesssim 1$. Finally, note that

$$\mathrm{diam}\, I = \sqrt{n}\ell(I) \overset{(2.29)}{<} \tau\sqrt{n} d_{\mathcal{F}}(I) \overset{(2.40)}{<} 3\sqrt{n}\tau r$$

and since I and I' touch, $\mathrm{dist}(x, I') \leq \mathrm{diam}\, I + r < (3\sqrt{n}\tau + 1)r$, so for $\tau > 0$ small enough, $I' \subseteq B(x, 2r)$. Thus, (2.44) follows.

Thus,

$$\begin{aligned}\mathcal{H}^d(\widehat{E}\cap B(x,r)) &\lesssim \sum_{I\in\mathscr{C}_1(x,r)} \ell(I)^d \lesssim \sum_{I\in\mathscr{C}_1(x,r)} \mathcal{H}^d_\infty(I'\cap Q_0)\\
&\leq \sum_{I'\in\mathscr{C}_1(x,r)} \sum_{i=1}^{N_0} \sum_{\substack{J\in\mathsf{Stop}(J_i)\\ J\subseteq I'}} (\operatorname{diam} J)^d\\
&\overset{(2.20)}{\lesssim} \sum_{I\in\mathscr{C}_1(x,r)} \sum_{i=1}^{N_0} \sum_{\substack{J\in\mathsf{Stop}(J_i)\\ J\subseteq I'}} \mu^{J_i}(J)\\
&\leq \sum_{I\in\mathscr{C}_1(x,r)} \sum_{i=1}^{N_0} \mu^{J_i}(I') \overset{(2.44)}{\lesssim} \sum_{i=1}^{N_0} \mu^{J_i}(B(x,2r)) \lesssim r^d.\end{aligned}$$

This proves (2.43).

□

LEMMA 2.20. *If $x\in\widehat{E}$ and $d_{\mathcal{F}}(x)<2r<8\ell(T)$, then*

$$\sum_{I\in\mathscr{C}_2(x,r)} \ell(I)^d \lesssim r^d. \tag{2.45}$$

PROOF. For $I\in\mathscr{C}_2(x,r)$, let $\tilde{Q}_I$ denote the child of Q_I' containing the center of Q_I'. We claim that the cubes $\{\tilde{Q}_I : I\in\mathscr{C}_2(x,r)\}$ have bounded overlap.

Indeed, suppose there were $I_1,....,I_\ell\in\mathscr{C}_2(x,r)$ distinct and a point

$$y\in\bigcap_{j=1}^{\ell}\tilde{Q}_{I_j}.$$

We can assume that $\tilde{Q}_{I_1}$ is the largest, and since they are all cubes, this implies $\tilde{Q}_{I_1}\supseteq\tilde{Q}_j$ for all j. Since

$$\operatorname{dist}(I_j,\tilde{Q}_{I_1}) \leq \operatorname{dist}(I_j,\tilde{Q}_{I_j}) \leq \operatorname{dist}(I_j,Q_{I_j}) \leq d_{\mathcal{F}}(I_j) \overset{(2.29)}{\lesssim} \tau^{-1}\ell(I_j) \tag{2.46}$$

and the I_j are disjoint, and because $\ell(I_j) \overset{(2.41)}{\sim}_\tau \ell(Q_{I_j}') \sim \ell(\tilde{Q}_{I_j})$, for given $\epsilon>0$, there can be at most boundedly many I_j (depending on ϵ and τ) for which $\operatorname{diam} I_j\geq\epsilon\ell(\tilde{Q}_{I_1})$. For the rest of the j, we have that

$$\operatorname{dist}(I_j,\tilde{Q}_{I_1}) \overset{(2.46)}{\lesssim} \tau^{-1}\ell(I_j) < \frac{\epsilon}{\tau}\ell(\tilde{Q}_{I_1}),$$

so for $\epsilon>0$ small enough, and recalling that $\lambda<2/c_0$ in Theorem 5.2 of the Appendix, this implies $I_j\subseteq c_0B_{Q_{I_j}'}$. Since $I_j\cap\widehat{T}\neq\emptyset$ and the balls $\{c_0B_Q : Q\in\mathcal{D}_k\}$ are disjoint for each k by Theorem 5.2, this means $\emptyset\neq I_j\cap Q_{I_j}'\subseteq I_j\cap Q_0$, and so $I_j\in\mathscr{C}_1(x,r)$, which is a contradiction since we assumed $I_j\in\mathscr{C}_2(x,r)$. Thus, there are no other j, and so $\ell\lesssim_\epsilon 1$. This finishes the proof that the sets $\{\tilde{Q}_I : I\in\mathscr{C}_2(x,r)\}$ have bounded overlap.

Fix $I\in\mathscr{C}_2(x,r)$ and let $J\in\mathscr{C}$ so that $J\cap\frac{c_0}{2}B_{\tilde{Q}_I}\neq\emptyset$. Then $\ell(J)<\tau d_{\mathcal{F}}(J)\leq\tau(1+c_0/2)\ell(\tilde{Q}_I)$; thus for τ small enough, $J\subseteq c_0B_{\tilde{Q}_I}$. Thus, if

$$\{J_I^i\}_{i=1}^{L_I} = \{J\in\mathscr{C} : J\cap\frac{c_0}{2}B_{\tilde{Q}_I}\neq\emptyset\},$$

(see Figure 2) since the $\tilde{Q}_I$ have bounded overlap, so do the cubes

$$\{J_I^i : i=1,...,L_I,\ \ I\in\mathscr{C}_2(x,r)\}.$$

For $I\in\mathscr{C}(x,r)$ and $i=1,...,L_I$,

$$\operatorname{dist}(I,J_I^i) \leq \operatorname{dist}(I,\tilde{Q}_I) \leq \operatorname{dist}(I,Q_I) \leq d_{\mathcal{F}}(I) < 2r,$$

hence $J_I^i\in\mathscr{C}_1(x,3r)$. Now we have by our assumptions that

$$d_{\mathcal{F}}(x) < 2r < 2\cdot(3r) = 3\cdot(2r) < 3\cdot 8\ell(T) = 24\ell(T).$$

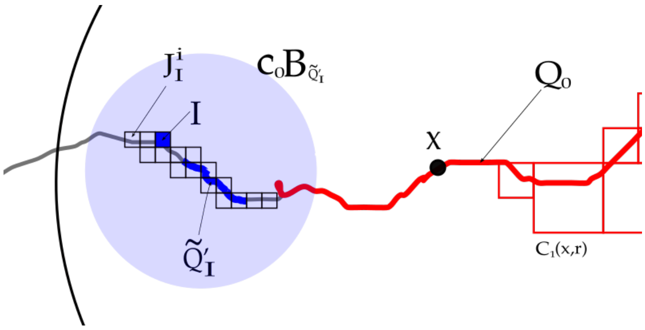

FIGURE 2. The dyadic cube $I \in \mathscr{C}_2(x,r)$ with its 'twin' cube $\tilde{Q}'_I \in \mathsf{Tree}(R)$ and the corresponding family of cubes $\{J_I^i\}$.

Thus, (2.43) holds for $3r$ in place of r, and so

$$\begin{aligned}\sum_{I\in\mathscr{C}_2(x,r)} \ell(I)^d &\sim \sum_{I\in\mathscr{C}_2(x,r)} \ell(\tilde{Q}_I)^d \sim_{c_1} \sum_{I\in\mathscr{C}_2(x,r)} \mathcal{H}^d_\infty\left(\frac{c_0}{2}B_{\tilde{Q}_I}\right)\\ &\lesssim \sum_{I\in\mathscr{C}_2(x,r)} \sum_{\substack{J\in\mathscr{C}\\ J\cap\frac{c_0}{2}B_{\tilde{Q}_I}\neq\emptyset}} \ell(J)^d = \sum_{I\in\mathscr{C}_2(x,r)}\sum_{i=1}^{L_I}\ell(J_I^i)^d\\ &\lesssim \sum_{J\in\mathscr{C}_1(x,3r)^d} \ell(J)^d \lesssim r^d\end{aligned}$$

where we used the bounded overlap property in the penultimate inequality.

□

Thus, combining the two previous lemmas, we have that

$$\mathcal{H}^d(\widehat{E}\cap B(x,r)) \lesssim \sum_{i=1}^{2}\sum_{I\in\mathscr{C}_i(x,r)} \ell(I)^d \lesssim r^d.$$

Now to complete the proof in this case, we need to show the reverse estimate. Let $I \in \mathscr{C}(x,r/2)$. Then (2.40) implies that for τ small enough, $I \subseteq B(x,r)$. Moreover, since $I\in\mathscr{C}$, $I\cap Q\neq\emptyset$ for some $Q\subseteq\widehat{T}$ with $\ell(Q)=\ell(T)$. If $I\in\mathscr{C}(x,r/2)$ is the cube so that $x\in\partial_d I$, then for τ small,

$$\operatorname{dist}(x,Q)\leq \operatorname{diam} I \overset{(2.40)}{\leq} 3\sqrt{n}\tau r < \frac{r}{4}$$

Thus, there is $y\in Q\cap B(x,r/4)$, and so we can find a subcube $Q'\subseteq B(x,r/2)\cap Q$ containing y so that $\ell(Q')\sim r$ and the cubes from $\mathscr{C}(x,r/2)$ cover Q'. Thus,

$$\begin{aligned}\mathcal{H}^d(B(x,r)\cap\widehat{E}) &\geq \sum_{I\in\mathscr{C}(x,r/2)} \mathcal{H}^d(\partial_d I) \sim \sum_{I\in\mathscr{C}(x,r/2)} \ell(I)^d\\ &\gtrsim \mathcal{H}^d_\infty(Q') \gtrsim \ell(Q')^d \sim r^d.\end{aligned}$$

Case 3: $2C_0\ell(T) > r > 4\ell(T)$.

Note that by the previous case,

$$\mathcal{H}^d(B(x,r)\cap 2B_T\cap\widehat{E}) \leq \mathcal{H}^d(2B_T\cap\widehat{E}) \lesssim \ell(T)^d \lesssim r^d.$$

So to prove upper regularity, we just need to verify

$$\mathcal{H}^d(B(x,r)\cap\widehat{E}\backslash 2B_T)\lesssim r^d.$$

If $I\cap B(x,r)\backslash 2B_T\neq\emptyset$, and if $y\in I\backslash 2B_T$,

$$\begin{aligned}\ell(I)\sim\tau d_{\mathcal{F}}(I) &\geq \tau(d_{\mathcal{F}}(y)-\operatorname{diam} I)\geq\tau\operatorname{dist}(y,T)-\tau\sqrt{n}\ell(I)\\ &\geq\tau\ell(T)-\tau\sqrt{n}\ell(I)\end{aligned}$$

and so for τ small enough,

$$\frac{\tau}{2}\ell(T)\leq\ell(I)$$

Moreover, since $I \cap B(x,r) \neq \emptyset$, $x \in \widehat{E}$, and $T \subseteq \bigcup_{J\in\mathscr{C}_{\mathcal{F}}} J$,

$$\begin{aligned}\ell(I) &< \tau d_{\mathcal{F}}(I) \leq \tau(\ell(T) + \operatorname{dist}(I,T))\\ &< \tau(\ell(T) + \operatorname{diam} I + 2r + \operatorname{dist}(x,T))\\ &< \tau(\ell(T) + \sqrt{n}\ell(I) + 4C_0\ell(T) + \operatorname{diam}\widehat{E})\\ &\lesssim \tau(\ell(T) + \ell(I))\end{aligned}$$

So for $\tau > 0$ small enough, we also have $\ell(I) \lesssim \tau\ell(T)$, hence $\ell(I) \sim \tau\ell(T)$. There can only be at most boundedly many disjoint cubes $I \in \mathscr{C}$ with $\ell(I) \sim \tau\ell(T)$, and so

$$\mathcal{H}^d(\widehat{E} \cap B(x,r)\backslash 2B_T) \lesssim \ell(T)^d \sim r^d.$$

For the lower bound, if $x \in \widehat{E} \cap 2B_T$, then $r > 4\ell(T)$ implies by the previous case that

$$\mathcal{H}^d(\widehat{E} \cap B(x,r)) \geq \mathcal{H}^d(\widehat{E} \cap 2B_T) \gtrsim \ell(T)^d \sim r^d.$$

Alternatively, if $x \in \widehat{E}\backslash 2B_T$, then by the arguments above, if $I \in \mathscr{C}$ contains x, then $\ell(I) \sim \tau\ell(T) \sim \tau r$, so for τ small enough, $I \subseteq B(x,r)$. Thus,

$$\mathcal{H}^d(\widehat{E} \cap B(x,r)) \geq \mathcal{H}^d(\partial_d I) \sim \ell(I)^d \sim r^d.$$

This completes the proof.

□

This finishes the proof of the Main Lemma.

4. A general lemma on quantitative properties

We now want to apply the approximation by Ahlfors regular sets obtained in the previous section to derive quantitative bounds on the sum of the β coefficients. The method we present is quite easy and general. The idea is the following: let us pick one of the quantitative properties described by David and Semmes. For example, the BAUP (which stands for bilateral approximation by union of planes) (see [DS93], II, Chapter 3), the GWEC (generalised weak exterior convexity) (see [DS93], II, Chapter 3), or the LS (local symmetry), see [DS91], Definition 4.2. On each cube $R \in \mathsf{Top}$, we run a stopping time on $\mathsf{Tree}(R)$ where we stop whenever we meet a cube which does not satisfy the chosen property. By doing so, we obtain a new tree and consequently a new approximating Ahlfors regular set. This time, however, this set will turn out to be uniformly rectifiable exactly because it approximates E at those scales where E is very well behaved.

Let us try to make all this precise.

Definition 2.21 (Quantitative property). By a quantitative property (QP) $\mathfrak{P}$ of E we mean a finite set of real numbers $\{p_1, ..., p_N\}$ with $0 < p_i \leq 1$ together with two subsets of $E \times \mathbb{R}_+ = E \times (0,\infty)$

$$\mathscr{G}^{\mathfrak{P}} = \mathscr{G}^{\mathfrak{P}}(p_1, ..., p_N) \text{ and } \mathscr{B}^{\mathfrak{P}} = \mathscr{B}^{\mathfrak{P}}(p_1, ..., p_N),$$

which depend on $\{p_1, ..., p_N\}$, such that

$$\mathscr{G}^{\mathfrak{P}} \cup \mathscr{B}^{\mathfrak{P}} = E \times \mathbb{R}_+ \text{ and } \mathscr{G}^{\mathfrak{P}} \cap \mathscr{B}^{\mathfrak{P}} = \emptyset. \tag{2.47}$$

We will call $\{p_1, ..., p_N\}$ the *parameters* of $\mathfrak{P}$.

If we want to specify the subset E upon which we are applying a quantitative property $\mathfrak{P}$, we may write, for example, $\mathscr{G}^{\mathfrak{P}}_E$, or $\mathscr{B}^{\mathfrak{P}}_E$. Let us give a few examples of quantitative properties described in the book [DS93]:

BWGL: The so-called 'Bilateral Weak Geometric Lemma' (BWGL) is a quantitative property. This appeared in Chapter 1; we recall its definition: given a real number $\epsilon > 0$, for each pair $(x,r) \in E \times \mathbb{R}_+$, BWGL asks whether there exists a plane P so that[8]

$$d_{B(x,r)}(E,P) < \epsilon.$$

If one such a plane exists, then we put $(x,r) \in \mathscr{G}^{BWGL}$; if not, then $(x,r) \in \mathscr{B}^{BWGL}$. This is clearly a partition of $E \times \mathbb{R}_+$. Hence BWGL is a QP with parameter ϵ.

[8]Recall the definition of $d_B(E,F)$ in (1.7).

LS: The 'Local Symmetry' (LS) property is defined as follows. Given $\epsilon > 0$, for each pair $(x,r) \in E \times \mathbb{R}_+$, we say $(x,r) \in \mathscr{B}^{LS}(\epsilon, \alpha)$ if there are $y, z \in B(x,r) \cap E$ so that $\operatorname{dist}(2y - z, E) \geq \epsilon r$.

LCV For the quantitative property 'Local Convexity' (LCV), we define $\mathscr{B}^{LCV}$ to be those $(x,r) \in E \times \mathbb{R}_+$ for which there are $y, z \in B(x,r) \cap E$ such that $\operatorname{dist}((y+z)/2, E) \geq \epsilon r$.

WCD: Let two positive numbers C_0 and ϵ be given. The 'Weak Constant Density' (WCD) condition asks the following: for $(x,r) \in E \times \mathbb{R}_+$, does a measure $\mu_{x,r}$ exists, such that

$$\operatorname{spt}(\mu_{x,r}) = E;$$
$$\mu_{x,r} \text{ is Ahlfors } d-\text{regular with constant } \mathsf{c}_0 \geq 1;$$
$$|\mu_{x,r}(y,t) - s^d| \leq \epsilon t^d \text{ for all } y \in E \cap B(x,r) \text{ and } 0 < s \leq r.$$

If one such a measure $\mu_{x,r}$ exists, then we put $(x,r) \in \mathscr{G}^{WCD}(\mathsf{c}_0^{-1}, \epsilon)$. If not, then $(x,r) \in \mathscr{B}^{WCD}(\mathsf{c}_0^{-1}, \epsilon)$. This is clearly a partition of $E \times \mathbb{R}_+$ and so WCD is a QP with parameters $(\mathsf{c}_0^{-1}, \epsilon)$.

BP: Let us give one more example. Let $1 \geq \theta > 0$ be a positive real number. The 'Big Projection' (BP) condition asks if for a pair (x,r), there exists a d-dimensional plane P such that

$$|\Pi_P(B(x,r) \cap E)| \geq \theta r^d,$$

where Π_P is the standard orthogonal projection onto P and $|\cdot|$ is the d-dimensional Lebesgue measure on P. We put $(x,r) \in \mathscr{G}^{BP}(\theta)$ if this is the case; otherwise $(x,r) \in \mathscr{B}^{BP}(\theta)$. Thus BP is a QP with parameter $\theta > 0$.

Definition 2.22. Fix a (small) parameter $\epsilon_1 > 0$ and two (large) constants $C_1, C_2 \geq 1$ and let $\mathfrak{P}$ be a quantitative property with parameters $\{p_1, ..., p_N\}$. We say that $\mathfrak{P}$ is (ϵ_1, C_1, C_2)-*continuous*, if there exist positive constants $0 < c_1, ..., c_N < \infty$ depending on ϵ_1 and C_1 such that the following holds. Let E_1 and E_2 be two subsets of $\mathbb{R}^n$ and let $B = B(x_B, r_B)$ be a ball so that

$$B \text{ is centered on } E_1;$$
$$(x_B, r_B) \in \mathscr{G}^{\mathfrak{P}}_{E_1}(p_1, ..., p_N);$$
$$d_{C_2 B}(E_1, E_2) < \epsilon.$$

If $B' = B(x_{B'}, r_{B'})$ is a ball so that

$$B' \text{ is centered on } E_2;$$
$$C_2 B' \subset B;$$
$$r_{B'} \geq \frac{r_B}{C_1},$$

then

$$(x_{B'}, r_{B'}) \in \mathscr{G}^{\mathfrak{P}}_{E_2}(c_1 p_1, ..., c_N p_N). \tag{2.48}$$

Remark 2.23. In particular a continuous quantitative property is *monotonic* (or *stable*) in the following sense; take a set E and a ball B centered on E with $(x_B, r_B) \in \mathscr{G}^{\mathfrak{P}}_E(p_1, ..., p_N)$. If we assume that $\mathfrak{P}$ is continuous and we take $E_1 = E_2 = E$ in Definition 2.22, then we see that $(x_{B'}, r_{B'}) \in \mathscr{G}^{\mathfrak{P}}_E(c_1 p_1, ..., c_N p_N)$ whenever $C_2 B' \subset B$ and $r_{B'} \geq \frac{r_B}{C}$.

Let us look at our concrete examples of QP, and see whether they are continuous, and thus stable.

- One can quite easily check that BWGL, LS, LCV, and BP are stable quantitative properties.
- On the other hand, the WCD is not.

Definition 2.24 (QP guaranteeing uniform rectifiability). We say a QP (with parameters $p_1, ..., p_N$) *guarantees uniform rectifiability* for Ahlfors d-regular sets with constant c_0 if, whenever A is Ahlfors d-regular with constant c_0 and

$$\mathbb{1}_{\mathscr{B}^{\mathfrak{P}}(p_1,...,p_N)} \frac{dr}{r} d\mathcal{H}^d|_A \text{ is a Carleson measure on } A \times R_+, \tag{2.49}$$

then A is a uniformly rectifiable set. Conversely, if A is uniformly rectifiable, then we say a QP (with parameters $p_1, ..., p_N$) is *guaranteed by uniform rectifiability* if the measure in (2.49) is a Carleson measure for the parameters $(p_1, ..., p_N)$.

Let us go back to our examples.

- In the two monographs [**DS91**] and [**DS93**], David and Semmes prove that the properties BWGL ([**DS93**], II.2, Proposition 2.2)[9], and WCD are indeed examples of QP guranteeing uniform rectifiability (see [**DS93**], I.2, Proposition 2.56, and [**To15**], Theorem 1.1). To further comment on the remark above, consider BWGL: if an Ahlfors d-regular set E is uniformly rectifiable, then there exists a universal constant $\epsilon_0 > 0$ so that for all $0 < \epsilon < \epsilon_0$, we have that
$$\int_{B\cap E} \int_0^R \mathbb{1}_{\mathscr{B}^{BWGL(\epsilon)}}(x,r) \, \frac{dr}{r} \, d\mathcal{H}^d(x) \leq C(\epsilon) r_B^d, \tag{2.50}$$
for all balls B centered on E with $r_B \leq \mathrm{diam}(E)$. In general, one may have that $C(\epsilon) \to \infty$ as $\epsilon \to 0$. On the other hand, it suffices to find a sufficiently small $\epsilon > 0$ for which (2.50) holds to prove that A is uniformly rectifiable.
- The property BP, on the other hand, does not guarantee uniform rectifiability. The standard 4-corner Cantor set is purely unrectifiable but still satisfy the Carleson measure condition above since it has large projections in some directions (although of course not *many* directions), see [**Dav06**, Part III Chapter 5].

Let now $\mathfrak{P}$ be a continuous quantitative property with parameters $\{p_1, ..., p_N\}$. For a cube $Q_0 \in \mathcal{D}$, we let[10]

$$\begin{aligned} \mathscr{B}^{\mathfrak{P}}(Q_0) &= \mathscr{B}^{\mathfrak{P}}(Q_0, p_1, ..., p_N) \\ &:= \left\{ Q \in \mathcal{D} \,|\, Q \subset Q_0; (z_Q, \ell(Q)) \in \mathscr{B}^{\mathfrak{P}} \right\}; \\ \mathscr{G}^{\mathfrak{P}}(Q_0) &= \mathscr{G}^{\mathfrak{P}}(Q_0, p_1, ..., p_N) := \mathcal{D}(Q_0) \setminus \mathscr{B}^{\mathfrak{P}}. \end{aligned} \tag{2.51}$$

Thus we put

$$\mathfrak{P}(Q_0, p_1, ..., p_N) := \mathfrak{P}(Q_0) := \sum_{Q \in \mathscr{B}^{\mathfrak{P}}_{Q_0}} \ell(Q)^d.$$

The following is the main result of this section. In later sections, we will show how the comparability results (Theorems 2.2 and 2.3) follow as corollaries.

Lemma 2.25. *Let $E \subset \mathbb{R}^n$ be a lower content (d, c_1)-regular set, and let*[11] $0 < \epsilon < 1$, *$C_2 \geq 1$, and $C_1 > 4C_2/\lambda$. There is C_0' depending on c_1 so that the following holds. Let $\mathfrak{P}$ be a QP of E with parameters $\{p_1, ..., p_N\}$ such that*

(2.52) *$\mathfrak{P}$ is (ϵ, C_1, C_2)-continuous with constants $c_1, ..., c_N$*

(2.53) *$\mathfrak{P}$ guarantees (and is guaranteed by) UR for C_0'-Ahlfors d-regular sets for parameters $c_1 p_1, ..., c_N p_N$;*

(2.54)

Then for any $Q_0 \in \mathcal{D}$

$$\beta_E(Q_0) \lesssim_{\mathsf{c}_1, C_1, \epsilon} \mathcal{H}^d(Q_0) + \mathfrak{P}(Q_0, c_1 p_1, ..., c_N p_N). \tag{2.55}$$

The proof of Lemma 2.25 will take up the rest of this section. Let us get started by first modifying the tree structure of $\mathsf{Top}(k_0)$, as in the statement of the Main Lemma by introducing a further stopping condition which is related to the QP $\mathfrak{P}$. Let $R \in \mathsf{Top}(k_0)$ and $R' \in \mathsf{Tree}(R)$. Let $\widetilde{\mathsf{Stop}}(R')$ be the maximal cubes in $\mathsf{Tree}(R)$ that are either in $\mathsf{Stop}(R)$ or contain a child in

[9] As was mentioned in Chapter 1.
[10] Recall from the Appendix, Theorem 5.2, that z_Q is the center of Q.
[11] Here λ is the parameter appearing in Appendix, Theorem 5.2.

$\mathscr{B}^{\mathfrak{P}}(Q_0)$, and let $\widetilde{\mathsf{Tree}}(R')$ be the subfamily of cubes $Q \in \mathsf{Tree}(R)$ contained in R' that are not properly contained in a cube from $\widetilde{\mathsf{Stop}}(R')$.

Let $\mathsf{N}ext_0(R) = \{R\}$ and for $j \geq 0$, if we have defined $\mathsf{N}ext_k(R)$, let

$$\mathsf{N}ext_{j+1}(R) = \bigcup_{R' \in \mathsf{N}ext_j} \bigcup_{Q \in \widetilde{\mathsf{Stop}}(R')} \mathsf{Child}(Q).$$

This process terminates at some integer K_R since $\mathsf{Tree}(R)$ is finite. Enumerate $\mathsf{N}ext_j = \{Q_i^j\}_{i=1}^{i_j}$.

Lemma 2.26. *Let $R \in \mathsf{Top}(k_0)$ and let $0 \leq j \leq K_R$ and $1 \leq i \leq i_j$. Then there exists a constant $c_1 < 1$ so that*

$$\sum_{Q \in \widetilde{\mathsf{Tree}}(Q_i^j)} \beta_E^{d,2}(3B_Q)\ell(Q)^d \leq C(c_1, \tau, n, C_0)\ell(Q_i^j)^d. \tag{2.56}$$

To prove Lemma 2.26, we will need the following Lemma from [AS18].

Lemma 2.27 ([AS18], Lemma 2.21). *Let $1 \leq p < \infty$ and E_1, E_2 lower content d-regular subsets of $\mathbb{R}^n$; let moreover $x \in E_1$ and choose a radius $r > 0$. Then if $y \in E_2$ is so that $B(x,r) \subset B(y, 2r)$, we have*

$$\beta_{E_1}^{p,d} \lesssim \beta_{E_2}^{p,d}(y, 2r) + \left(\frac{1}{r^d} \int_{E_1 \cap B(x,2r)} \left(\frac{\mathrm{dist}(y, E_2)}{r} \right)^p \, d\mathcal{H}^d_\infty(y) \right)^{\frac{1}{p}}. \tag{2.57}$$

Let R, j, i as above. Let $E_{Q_i^j} = E_{i,j}$ be the Ahlfors regular set obtained from the Main Lemma for $\widetilde{\mathsf{Tree}}(Q_i^j)$ and $d_{Q_i^j}$ be the function defined in (2.6), where, in this instance, $\mathcal{F} = \widetilde{\mathsf{Stop}}(Q_i^j)$ and $\mathsf{Str} = \widetilde{\mathsf{Tree}}(Q_i^j)$. Specifically, for $C_0 > 4$, as in (2.36), we set

$$\hat{T}_{i,j} := \left\{ Q \in \mathcal{D} \,|\, \ell(Q) = \ell(Q_i^j),\, Q \cap C_0 B_{Q_i^j} \neq \emptyset \right\};$$

following (2.37), we then put

$$\mathscr{C}_{i,j} := \left\{ I \in \mathscr{I} \,|\, I \cap \hat{T} \neq \emptyset \text{ and } I \text{ is maximal with } \ell(I) < \tau d_{Q_i^j}(I) \right\}.$$

Then

$$E_{i,j} := \bigcup_{I \in \mathscr{C}_{i,j}} \partial_d I. \tag{2.58}$$

It follows from the Main Lemma that $E_{i,j}$ is Ahlfors d-regular.

Lemma 2.28. *Let $k_0, \tau > 0, R, j$ and i as above. Then $E_{i,j}$ is uniformly rectifiable.*

We want to use the fact that $\mathfrak{P}$ guarantees uniform rectifiability and that it is continuous. We will show that there exist constants

$$c_1, \ldots, c_N$$

such that the measure

$$\mathbb{1}_{\mathscr{B}^{\mathfrak{P}}(c_1 p_1, \ldots, c_N p_N)}(x, r) \frac{dt}{t} d\mathcal{H}^d(x) \tag{2.59}$$

is Carleson on $E_{i,j} \times \mathbb{R}_+$. We test this measure on a ball B centered on $E_{i,j}$ and with radius r_B. Let $\eta > 0$ be a positive constant to be fixed soon. Note that

$$\int_{B \cap E_{i,j}} \int_0^{\eta \tau d_{Q_i^j}(x)} \mathbb{1}_{\mathscr{B}^{\mathfrak{P}}(c_1 p_1, \ldots, c_N p_N)}(x, r) \, \frac{dr}{r} d\mathcal{H}^d(x) \lesssim_{n,d} r_B^d. \tag{2.60}$$

holds automatically: indeed, for any $x \in E_{i,j}$ and whenever $0 < r \leq \eta \tau d_{Q_i^j}(x)$ (if $\eta < 1/10$, say), $B(x,r) \cap E_{i,j}$ is just a finite union of d-dimensional planes, and the number of planes in this union is bounded above by a universal constant only depending on n and d. Therefore $B(x,r) \cap E_{i,j}$ is uniformly rectifiable and thus (2.60) holds. Also, using the Ahlfors regularity of $E_{i,j}$, it is immediate to see that

$$\int_{B \cap E_{i,j}} \int_{\eta \tau d_{Q_i^j}(x)}^{\eta^{-1} d_{Q_i^j}(x)} \mathbb{1}_{\mathscr{B}^{\mathfrak{P}}(c_1 p_1, \ldots, c_N p_N)}(x, r) \, \frac{dr}{r} d\mathcal{H}^d(x) \lesssim_{\tau, \eta} r_B^d. \tag{2.61}$$

Let us check that

$$\int_{B\cap E_{i,j}} \int_{\eta^{-1} d_{Q_i^j}(x)}^{\frac{\tau}{10}\ell(Q_i^j)} \mathbb{1}_{\mathscr{B}^{\mathfrak{P}}(c_1p_1,\dots,c_Np_N)}(x,r)\,\frac{dr}{r}\,d\mathcal{H}^d(x) \lesssim_{\tau,\eta} r_B^d. \tag{2.62}$$

LEMMA 2.29. *Let $(x,r)\in E_{i,j}\times\mathbb{R}_+$ be such that*

$$\eta^{-1} d_{Q_i^j}(x) \leq r \leq \tau\ell(Q_i^j). \tag{2.63}$$

Then, for $\eta>0$ sufficiently small (depending only on n), there exists a cube P in $\widetilde{\mathsf{Tree}}(Q_i^j)$ so that

$$B_P \subset B(x,r).$$

PROOF. For this proof, we put $Q=Q_i^j$. Let I_x be the cube in $\mathscr{C}_Q$ containing x, so $\ell(I_x)\sim\tau d_Q(x)$. Let P^* be the minimiser of $d_Q(x)$. Note that

$$\operatorname{dist}(x,P^*) \leq d_Q(x) \leq \eta r. \tag{2.64}$$

Let us look at two distinct cases.

Case 1. Suppose first that

$$d_Q(x) = \ell(P^*) + \operatorname{dist}(x,P^*) \leq 2\ell(P^*). \tag{2.65}$$

Then we immediately obtain that

$$\ell(P^*) \leq d_Q(x) \leq 2\ell(P^*)$$

and therefore that

$$\ell(P^*) \sim \tau^{-1}\ell(I_x). \tag{2.66}$$

But (2.65) also implies that

$$\operatorname{dist}(x,P^*) \leq \ell(P^*) \tag{2.67}$$

Now, because of the assumption (2.63), we see that (using also (2.66))

$$r \geq \eta^{-1} d_Q(x) \geq \eta^{-1} d_Q(I_x) \sim \eta^{-1}\tau^{-1}\ell(I_x) \sim \eta^{-1}\ell(P^*),$$

and so, because (2.67) and (2.64), we have for η small $B_{P^*}\subset B(x,r)$.

Case 2. Suppose now that

$$d_Q(x) = \ell(P^*) + \operatorname{dist}(x,P^*) \leq 2\operatorname{dist}(x,P^*).$$

Then we have

$$\operatorname{dist}(x,P^*) \sim d_Q(x) \leq C\eta r.$$

Also, by (2.63), it holds that

$$\ell(P^*) \leq d_Q(x) \leq \eta r.$$

This implies, for $\eta>0$ sufficiently small, that also in this case we have $B_{P^*}\subset B(x,r)$. □

LEMMA 2.30. *There exist constants $(c_1,\dots,c_N)$ such that the following holds. Let $(x,r)\in E_{i,j}\times\mathbb{R}_+$ be such that*

$$\eta^{-1} d_{Q_i^j}(x) \leq r \leq \tau\ell(Q_i^j). \tag{2.68}$$

Then

$$(x,r)\in\mathscr{G}^{\mathfrak{P}}_{E_{i,j}}(c_1p_1,\dots,c_Np_N).$$

PROOF. Let C_1 and C_2 be the two constants from the statement of Lemma 2.25. We know from Lemma 2.29 that if (x,r) satisfies (2.68), then there exists a cube $P^*\in\widetilde{\mathsf{Stop}}(Q_i^j)$ such that $B_{P^*}\subset B(x,r)$. Thus, there must exist an ancestor $\widehat{P}^*\in\widetilde{\mathsf{Tree}}(Q_i^j)$ of P^* so that

$$\lambda\ell(\widehat{P}^*) \leq 4C_2 r < \ell(\widehat{P}^*), \tag{2.69}$$

and thus so that $B(x,C_2r)\subset B_{\widehat{P}^*}$, and since $C_1>4C_2/\lambda$, we also have $r\geq\ell(\widehat{P}^*)/C_1$. But recall that if $\widehat{P}^*\in\widetilde{\mathsf{Tree}}(Q_i^j)$, then we must have, by definition, that $(z_{\widehat{P}^*},\ell(\widehat{P}^*))\in\mathscr{G}^{\mathfrak{P}}(p_1,\dots,p_N)$.

Let us check that

$$d_{C_2 B_{\widehat{P}^*}}(E_{i,j}, E) < \tau.$$

By (2.8), if $y \in E \cap C_2 B_{\widehat{P}^*}$

$$\operatorname{dist}(y, E_{i,j}) \lesssim \tau d_{Q_{i,j}}(y) \le \tau(\operatorname{dist}(y, \widehat{P}^*) + \ell(\widehat{P}^*)) \lesssim C_2 \tau \ell(\widehat{P}^*).$$

That for any $x \in E_{i,j} \cap C_2 B_{\widehat{P}^*}$ we have $\operatorname{dist}(x, E) \lesssim \tau \ell(\widehat{P}^*)$ follows in the same way, since any such x is contained in a dyadic cube I touching E so that

$$\ell(I) < \tau d_{Q_i^j}(I) \overset{(2.68)}{\le} \eta \tau r \overset{(2.69)}{\le} 8\eta\tau\ell(\widehat{P}^*).$$

Choosing τ in the construction of $\mathscr{C}_{i,j}$ appropriately (depending on ϵ and C_2), the lemma follows from the (ϵ, C_1, C_2)-continuity of $\mathfrak{P}$. $\square$

Proof of Lemma 2.28. We have shown that there exist constants $c_1, ..., c_N$ such that, for any pair $(x, r) \in E_{i,j} \times \mathbb{R}_+$ with

$$\eta^{-1} d_{Q_i^j}(x) \le r \le \tau \ell(Q_i^j)$$

we have

$$(x, r) \in \mathscr{G}^{\mathfrak{P}}_{E_{i,j}}(c_1 p_1, ..., c_N p_N). \tag{2.70}$$

Thus the integral in (2.62) equals to zero. Now, we also see that, trivially

$$\int_{B \cap E_{i,j}} \int_{\frac{\tau}{10}\ell(Q_i^j)}^{\operatorname{diam}(E_{i,j})} \mathbb{1}_{\mathscr{B}^{\mathfrak{P}}(c_1 p_1, ..., c_N p_N)}(x, r) \, \frac{dr}{r} d\mathcal{H}^d(x) \lesssim_\tau r_B^d. \tag{2.71}$$

This together with the previous estimates (2.60), (2.61) and (2.62) proves that the measure $\mathbb{1}_{\mathscr{B}^{\mathfrak{P}}(c_1 p_1, ..., c_N p_N)}(x, r) \frac{dr}{r} d\mathcal{H}^d|_{E_{i,j}}(x)$ is a Carleson measure on $E_{i,j} \times \mathbb{R}_+$; then, because $\mathfrak{P}$ guarantees uniform rectifiability with the appropriate parameters and it is (ϵ, C_1, C_2)-continuous, $E_{i,j}$ is uniformly rectifiable. Note that all the constants involved depend only on n, d, τ, η (and c_1); in particular, they are all independent of Q_i^j, R and k_0. $\square$

Proof of Lemma 2.26. We want to apply Lemma 2.27 with $E_1 = E$, $E_2 = E_{i,j}$ and $p = 2$. For $Q \in \widetilde{\mathsf{Tree}}(Q_i^j)$, recall that z_Q denotes the center of Q. By (2.8), we know that $\operatorname{dist}(x_Q, E_{i,j}) \lesssim \tau d_{Q_i^j}(x_Q) \le \tau \ell(Q)$, and in particular, if we denote by x'_Q the point in $E_{i,j}$ which is closest to x_Q, we see that $B_Q := B(z_Q, \ell(Q)) \subset B(x'_Q, 2\ell(Q)) =: B'_Q$ for τ small enough. Hence for each cube $Q \in \mathsf{Tree}(Q_i^j)$ the hypotheses of Lemma 2.27 are satisfied and we may write

$$\begin{aligned} \sum_{Q \in \widetilde{\mathsf{Tree}}(Q_i^j)} \beta_E^{2,d}(3B_Q)^2 \ell(Q)^d &\lesssim \sum_{Q \in \widetilde{\mathsf{Tree}}(Q_i^j)} \beta_{E_{i,j}}^{2,d}(6B'_Q)^2 \ell(Q)^d \\ &+ \sum_{Q \in \widetilde{\mathsf{Tree}}(Q_i^j)} \left(\frac{1}{\ell(Q)^d} \int_{6B_Q \cap E} \left(\frac{\operatorname{dist}(x, E_{i,j})}{\ell(Q)} \right)^2 d\mathcal{H}^d_\infty(x) \right) := I_1 + I_2. \end{aligned}$$

We first look at I_1. We apply Theorem 5.2 from the Appendix (that is, the existence of Christ-David cubes), to $E_{i,j}$; let us denote the cubes so obtained by $\mathcal{D}_{E_{i,j}}$. Note that for each $P \in \widetilde{\mathsf{Tree}}(Q_i^j)$ with $P \in \mathcal{D}(k_0)$, x'_P belongs to some cube $P' \in \mathcal{D}_{E_{i,j}}$ so that $\ell(P') \sim \ell(P)$; hence there exists a constant $C_3 \ge 1$ so that

$$6B'_P \subset C_3 B_{P'}. \tag{2.72}$$

This in turn implies that $\beta_{E_{i,j}}^{p,d}(6B'_P) \lesssim_{p,n,d,C_3} \beta_{E_{i,j}}^{p,d}(C_3 B_{P'})$. Hence,

$$\sum_{\substack{P \in \mathsf{Tree}(Q_i^j) \\ P \in \mathcal{D}(k_0)}} \beta_{E_{i,j}}^{2,d}(6B'_P)^2 \ell(P)^d \lesssim_{p,n,d,C_3} \sum_{\substack{P' \in \mathcal{D}_{E_{i,j}} \\ \ell(P') \lesssim \ell(Q_i^j)}} \beta_{E_{i,j}}^{2,d}(C_3 B_{P'})^2 \ell(P')^d. \tag{2.73}$$

Since $E_{i,j}$ is uniformly rectifiable, we immediately have that $I_1 \lesssim \ell(Q_i^j)^d$ by the main results of [**DS91**] (in particular, see (C3) and (C6) in [**DS91**, Chapter 1])[12].

[12] This is Theorem 1.2 in Chapter 1.

Let us now worry about I_2. We put

$$\mathsf{Approx}(Q_i^j) := \Big\{ \text{maximal } S \in \mathcal{D}(k_0) \,|\, \text{there is } I \in \mathscr{C}_{i,j} \text{ s.t. } I \cap S \neq \emptyset \tag{2.74}$$
$$\text{and } \lambda\ell(S) \leq \ell(I) \leq \ell(S) \Big\}.$$

It is clear that

$$Q_i^j \subset \bigcup_{S \in \mathsf{Approx}(Q_i^j)} S, \tag{2.75}$$

since $\mathscr{C}_{i,j}$ covers Q_i^j. Now let $x \in Q_i^j$. We claim that there exists a cube $S \in \mathsf{Approx}(Q_i^j)$ so that

$$\operatorname{dist}(x, E_{i,j}) \leq C\ell(S). \tag{2.76}$$

By (2.75), we see that if $x \in Q_i^j$, then there exists an $S \in \mathsf{Approx}(Q_i^j)$ so that $x \in S$. But, then, by definition, there exists an $I \in \mathscr{C}_{i,j}$ such that $\ell(I) \leq \ell(S)$ and $I \cap S \neq \emptyset$. Thus

$$\operatorname{dist}(x, E_{i,j}) \leq \operatorname{diam} I + \operatorname{dist}(x, I) \lesssim \ell(S).$$

We now estimate I_2 as follows: first,

$$\begin{aligned}\frac{1}{\ell(Q)^d} \int_{6B_Q \cap E} \left(\frac{\operatorname{dist}(x, E_{i,j})}{\ell(Q)} \right)^2 d\mathcal{H}^d_\infty(x) &\overset{(2.76)}{\lesssim} \sum_{\substack{S \in \mathsf{Approx}(Q_i^j) \\ S \cap 6B_Q \neq \emptyset}} \int_S \frac{\ell(S)^2}{\ell(Q)^{d+2}}\, d\mathcal{H}^d_\infty \\ &\lesssim \sum_{\substack{S \in \mathsf{Approx}(Q_i^j) \\ S \cap 6B_Q \neq \emptyset}} \frac{\ell(S)^{2+d}}{\ell(Q)^{2+d}}. \end{aligned} \tag{2.77}$$

Hence we obtain that

$$\begin{aligned} I_2 &\overset{(2.77)}{\lesssim} \sum_{Q \in \widetilde{\mathsf{Tree}}(Q_i^j)} \sum_{\substack{S \in \mathsf{Approx}(Q_i^j) \\ S \cap 6B_Q \neq \emptyset}} \frac{\ell(S)^{d+2}}{\ell(Q)^2} \\ &\lesssim \sum_{\substack{S \in \mathsf{Approx} Q_i^j \\ S \cap 6B_{Q_i^j}}} \ell(S)^{d+2} \sum_{\substack{Q \in \widetilde{\mathsf{Tree}}(Q_i^j) \\ S \cap 6B_Q \neq \emptyset}} \frac{1}{\ell(Q)^2}. \end{aligned} \tag{2.78}$$

Note that the number of cubes $Q \in \widetilde{\mathsf{Tree}}(Q_i^j)$ which belong to a given generation and such that $S \cap 6B_Q \neq \emptyset$ is bounded above by a constant C which depends on n. Indeed, if $S \cap 6B_Q \neq \emptyset$, then we must have that $\operatorname{dist}(Q, S) \leq 6\ell(Q)$. Moreover, because $S \in \mathsf{Approx}(Q_i^j)$ and using Lemma 2.11, we see that, if $I \in \mathscr{C}_{i,j}$ is so that $I \cap S \neq \emptyset$ and $\ell(S) \sim \ell(I)$ (as in (2.74)),

$$\ell(S) \sim \ell(I) \sim \tau d_{Q_i^j}(I) \lesssim \tau(\ell(Q) + \operatorname{dist}(I, Q)) \leq \tau\ell(Q) + \tau(6\ell(Q) + 2\ell(S))$$

so for τ small enough, $\ell(S) \lesssim \ell(Q)$. Thus we can sum the interior sum in (2.78):

$$\sum_{\substack{Q \in \mathsf{Tree}(Q_i^j) \\ S \cap 6B_Q \neq \emptyset}} \frac{1}{\ell(Q)^2} \lesssim_\tau \frac{1}{\ell(S)^2}.$$

Finally, we see that

$$I_2 \lesssim_{\tau,n} \sum_{\substack{S \in \mathsf{Approx}(Q_i^j) \\ S \cap 6B_{Q_i^j}}} \frac{\ell(S)^{d+2}}{\ell(S)^2} = \sum_{\substack{S \in \mathsf{Approx}(Q_i^j) \\ S \cap 6B_{Q_i^j}}} \ell(S)^d. \tag{2.79}$$

Now, by definition of $\mathsf{Approx}(Q_i^j)$, the last sum in (2.79) is bounded above by a constant times

$$\sum_{I \in \mathscr{C}_{i,j}} \ell(I^d) \lesssim \mathcal{H}^d \left(\bigcup_{I \in \mathscr{C}_{i,j}} \partial_d I \right) = \mathcal{H}^d(E_{i,j}) \lesssim \ell(Q_i^j)^d,$$

where we also used the Ahlfors regularity of $E_{i,j}$. This proves (2.56). $\square$

Proof of Lemma 2.25. Let $Q_0 \in \mathcal{D}$ as in the statement of the Lemma. Then we see that

$$\sum_{R\in\mathsf{Top}(k_0)} \sum_{j=0}^{K_R} \sum_{Q\in\mathsf{N}ext_j(R)} \sum_{P\in\widetilde{\mathsf{Tree}}(Q)} \beta_E^{2,d}(3B_P)^2\ell(P)^d$$

$$\overset{(2.56)}{\lesssim_\tau} \sum_{R\in\mathsf{Top}(k_0)} \sum_{j=0}^{K_R} \sum_{Q\in\mathsf{N}ext_j(R)} \ell(Q)^d. \tag{2.80}$$

Note that for $1 \leq j \leq K_R$, if $Q \in \mathsf{N}ext_j(R)$, then there is a sibling Q' of Q so that $(z_{Q'}, \ell(Q')) \in \mathscr{B}^{\mathfrak{P}}$. Also recall that we put $\mathsf{N}ext_0(R) = \{R\}$. Then any cube appearing in the sum (2.80), either belongs to $\mathsf{Top}(k_0)$ (whenever it belongs to $\mathsf{N}ext_0(R)$), or is adjacent to a cube in $\mathscr{B}^{\mathfrak{P}}(Q_0, p_1, ..., p_N)$, as defined in (2.51). Thus we see that

$$(2.80) \lesssim_\tau \left(\sum_{R\in\mathsf{Top}(k_0)} \ell(R)^d + \mathfrak{P}(Q_0, p_1, ..., p_N)\right) \overset{(2.5)}{\lesssim_\tau} \mathcal{H}^d(Q_0) + \mathfrak{P}(Q_0).$$

Note that all these estimates were independent of k_0. Sending k_0 to infinity and recalling (2.91) (and recalling that $\ell(Q_0)^d \lesssim_c \mathcal{H}^d(Q_0)$) gives the estimate (2.55). □

5. Applications: The dimensionless quantities LS and LCV

Here we give a proof of Theorem 2.3

Proof. First, it is not hard to show that there is $c > 0$ so that if $Q \in \mathscr{G}_E^{BWGL}(Q_0, c\epsilon)$, then for any children Q' of Q, since

$$\ell(Q') = \rho\ell(Q) < \frac{1}{4}\ell(Q),$$

we have $Q' \in \mathscr{G}_E^{LS}(\epsilon)$. Using this fact, we get

$$\mathcal{H}^d(R) + \mathrm{LS}(R, \epsilon) \leq \mathcal{H}^d(R) + \mathrm{BWGL}(R, c\epsilon) \lesssim \beta_E(R)$$

and so we just need to prove the reverse inequality.

First we show that for all $C > 1$ and $\epsilon > 0$ is small depending on C and $B \in \mathscr{G}_E^{LS}(\epsilon)$ and E' is another lower d-regular set so that $d_{4B}(E, E') < \epsilon$, then any ball B' with $4B' \subseteq B$ centered on E' with $r_{B'} \geq r_B/C$, we have that $B' \in \mathscr{G}_{E'}^{LS}(c\epsilon)$ for some $c > 0$, and so LS is $(\epsilon, C, 4)$-continuous for all $C > 1$ and $\epsilon > 0$ sufficiently small depending on C.

Let $x', y' \in E' \cap B'$, then there are $x, y \in E$ with $|x - x'|, |y - y'| < 4\epsilon r_B$. For $\epsilon > 0$ small depending on C, since $r_{B'} \geq r_B/C$, $x, y \in \frac{3}{2}B'$, and so $2x - y \in 3B' \subseteq B$. Since $B \in \mathscr{G}^{\mathrm{LS}}(\epsilon)$, there is $\xi \in E$ so that $|2x - y - \xi| < \epsilon r_B$. For $\epsilon > 0$ small enough, since $2x - y \in \frac{3}{2}B'$, $\xi \in 4B' \subseteq B$, thus there is $\xi' \in E'$ with $|\xi - \xi'| < 4\epsilon r_B$. Thus,

$$\mathrm{dist}(2x' - y', E') \leq |2x' - y' - \xi'| \leq |2x - y - \xi| + |x - x'| + |y - y'| + |\xi - \xi'| < 16\epsilon r_B.$$

Hence, $B' \in \mathscr{G}_{E'}^{\mathrm{LS}}(16\epsilon)$. Thus, for $\epsilon > 0$ small enough, Lemma 2.25 implies the second half of (2.81). This completes the proof.

□

Another dimensionless quantity is the LCV. This can be proven in much the same way, so we omit the proof.

Theorem 2.31. *Let $E \subseteq \mathbb{R}^n$ be a lower (d, c_1)-regular set and $\mathcal{D}$ its Christ-David cubes. Then for $\epsilon > 0$ small enough, and $R \in \mathcal{D}$,*

$$\beta_E(R) \sim \mathcal{H}^d(R) + \mathrm{LCV}(R, \epsilon). \tag{2.81}$$

6. Application: the BAUP

In this section, we show that we can apply Lemma 2.25 to the quantitative property BAUP (recall the definition (1.9)). Namely, we will show that BAUP is (ϵ, C_1, C_2)-continuous. That BAUP guarantees rectifiability is due to David and Semmes, see [**DS93**], Proposition 3.18.

Let $\epsilon_0 > 0$ and $\mathsf{C}_0 \geq 1$ be given. Let us first define the actual partition that BAUP determines. We put

$$\mathscr{G}^{\mathrm{BAUP}}(\epsilon_0, \mathsf{C}_0) = \mathscr{G}^{BAUP} := \Big\{ (x,r) \in E \times R_+ \,|\, \text{there is a family } \mathcal{F} \text{ of } d\text{-planes}$$

$$\text{s.t. } d_{B(x,\mathsf{C}_0 r)}(E, \cup_{P \in \mathcal{F}} P) < \epsilon_0 \Big\}$$

$$\mathscr{B}^{\mathrm{BAUP}}(\epsilon_0, \mathsf{C}_0) = \mathscr{B}^{BAUP} := E \times \mathbb{R}_+ \setminus \mathscr{G}^{\mathrm{BAUP}}.$$

LEMMA 2.32. *Let $\epsilon_0 > 0$, $\mathsf{C}_0 \geq 1$, and consider the quantitative property BAUP with parameters $(\epsilon_0, \mathsf{C}_0)$. If $C_1 \geq 1$, $C_2 > 2\mathsf{C}_0$, ϵ_0 is small enough (depending on C_2 and C_1), and $0 < \epsilon_1 \leq \epsilon_0$ then BAUP is (ϵ_1, C_1, C_2)-continuous.*

PROOF. Let us consider two subsets E_1, E_2 or $\mathbb{R}^n$. From Definition 2.22, we take a ball $B = B(x_B, r_B)$ centered on E_2 and so that, first,

$$(x_B, r_B) \in \mathscr{G}^{BAUP}_{E_1}(\epsilon_0, \mathsf{C}_0),$$

and second,

$$d_{C_2 B}(E_1, E_2) < \epsilon_1, \tag{2.82}$$

where C_2 and $\epsilon_1 \leq \epsilon_0$ will be determined later with respect to C_0 and ϵ_0. Thus, there is a union of d-dimensional planes $\mathcal{F}$ so that

$$d_{\mathsf{C}_0 B}(E_1, \mathcal{F}) < \epsilon_0.$$

Next, we consider a ball $B' = B(x'_B, r'_B)$ centered this time on E_2 with $C_2 B' \subseteq B$ and so that $r'_B \geq \frac{r_B}{C_1}$. We want to show that for any such a ball B',

$$d_{\mathsf{C}_0 B'}(E_2, \mathcal{F}) < c_1 \epsilon_0 r'_B. \tag{2.83}$$

for some constant c_1 to be determined. Let $y \in E_2 \cap \mathsf{C}_0 B'$. Since $2B' \subseteq C_2 B' \subset B$, we have $2\mathsf{C}_0 B' \subseteq \mathsf{C}_0 B \subseteq C_2 B$, so we can use (2.82) to find an $x \in E_1$ so that $|x - y| < \epsilon_1 C_2 r_B$. Since $\epsilon_0 \leq 1$, $x \in E_1 \cap 2\mathsf{C}_0 B' \subseteq \mathsf{C}_0 B$, and because $(x_B, r_B) \in \mathscr{G}^{BAUP}_{E_1}(\epsilon_0, \mathsf{C}_0)$, it holds that $\mathrm{dist}(x, \mathcal{F}) < \epsilon_0 \mathsf{C}_0 r_B$. Now, because $\epsilon_1 \leq \epsilon_0$, we have that

$$\sup_{y \in E_2 \cap \mathsf{C}_0 B'} \mathrm{dist}(y, \mathcal{F}) \leq \epsilon_1 C_2 r_B + \epsilon_0 \mathsf{C}_0 r_B \leq (2C_2 C_1)\, \epsilon_0 r'_B$$

Next, for $q \in \mathcal{F} \cap \mathsf{C}_0 B'$, we look at $\mathrm{dist}(q, E_2)$; note in particular that $q \in \mathcal{F} \cap \mathsf{C}_0 B$ and thus, because $d_{\mathsf{C}_0 B}(E_1, \mathcal{F}) < \epsilon_0$, there is an $x \in E_1$ with $|x - q| \leq \epsilon_0 \mathsf{C}_0 r_B$. Moreover, choosing $C_2 > 2\mathsf{C}_0$, since $\epsilon_0 \leq 1$, we also have that $x \in 2\mathscr{C}_0 B \subseteq C_2 B$, and thus $\mathrm{dist}(x, E_2) < C_2 \epsilon_1 r_B$. All in all, we obtain that

$$\begin{aligned}\sup_{q \in E_2 \cap \mathsf{C}_0 B'} \mathrm{dist}(q, E_2) &\leq |x - q| + \mathrm{dist}(x, E_2) \\ &\leq \mathsf{C}_0 \epsilon_0 r_B + C_2 \epsilon_1 r_B \\ &\leq (2C_2 C_1)\, \epsilon_0 r'_B.\end{aligned}$$

This implies (2.83) with $c_1 = 2C_1 C_2$; thus BAUP is (ϵ_1, C_1, C_2)-continuous, whenever $\epsilon_1 \leq \epsilon_0$, and C_2 is sufficiently large, with respect to the parameter C_0. □

We can now prove Theorem 2.2. Firstly, note that we immediately have

$$\mathrm{BAUP}(Q_0, \mathsf{C}_0, \epsilon) \leq \mathrm{BWGL}(Q_0, \mathsf{C}_0, \epsilon) \lesssim \beta_E(Q_0).$$

Furthermore, since $\mathrm{BAUP}(\mathscr{C}_0, \epsilon)$ guarantees and is guaranteed by UR for all $\epsilon > 0$ sufficiently small depending on C_0 by [DS93, Theorem III.3.18]. Since it is also (ϵ, C_1, C_2)-continuous for $C_2 > 2\mathscr{C}_0$ and all $C_1 \geq 1$ and $\epsilon > 0$ sufficiently small, we have, for all $\mathscr{C}_0 \geq 1$ and ϵ small enough (depending on $\mathscr{C}_0$)

$$\beta_E(Q) \lesssim \mathcal{H}^d(Q_0) + \mathrm{BAUP}(Q_0, \mathsf{C}_0, \epsilon).$$

7. Application: the GWEC

Let us give one last example of quantitative property which can be handled within the framework of Lemma 2.25. For a parameter $\epsilon_0 > 0$, we put in $\mathscr{B}^{GWEC}$ all the pairs $(x, r) \in$

$E \times R_+$ for which there exists an $(n-d-1)$-dimensional sphere S satisfying the following three conditions.

$$S \subset B(x,r) \text{ and } \operatorname{dist}(S,E) > \epsilon_0 r; \tag{2.84}$$

$$S \text{ can be contracted to a point inside}$$
$$\{y \in B(x,r) \mid \operatorname{dist}(y,E) > \epsilon_0 r\}\,; \tag{2.85}$$

$$\operatorname{ch}(S) \cap E \neq \emptyset, \tag{2.86}$$

where $\operatorname{ch}(S)$ is the convex hull of S. We then put

$$\mathscr{G}^{GWEC}(\epsilon_0) := E \times \mathbb{R}_+ \setminus \mathscr{B}^{GWEC}(\epsilon_0).$$

We want to check that we can apply Lemma 2.25 with this quantitative property. That the GWEC guarantees uniform rectifiability is Theorem 3.28 in [**DS93**]. All that's left to do is to prove that GWEC is continuous.

LEMMA 2.33. *The quantitative property GWEC with parameter $\epsilon_0 > 0$ is (ϵ_1, C_1, C_2)-continuous, for all $C_1 \geq 3$, for all $C_2 \geq 1$ and whenever ϵ_1 is sufficiently small with respect to $\epsilon_0, C_1,$ and C_2.*

PROOF. Let E_1 and E_2 be two subsets of $\mathbb{R}^n$. Let $B = B(x_B, r_B)$ be a ball centered on E_1 so that $(x_B, r_B) \in \mathscr{G}^{GWEC}_{E_1}(\epsilon_0)$ and

$$d_{C_2 B}(E_1, E_2) < \epsilon_1 C_2 r_B. \tag{2.87}$$

We want to find a constant c_1 so that, for any ball $B' = B(x'_B, r'_B)$ centered on E_2 and with $2B' \subset B$ and $r'_B \geq r_B/C_1$, we have that $(x'_B, r'_B) \in \mathscr{G}^{GWEC}_{E_2}(c_1\epsilon_0)$.

We argue by contradiction. Suppose that for some c_1 (to be determined), we can find a sphere S' as in (2.84), (2.85) and (2.86) for the ball B'. We will construct a sphere S for B satisfying the same three conditions: this will contradict the hypothesis that B is a good ball.

Let $\hat{y} \in E_2 \cap \operatorname{ch}(S')$; note that in particular $\hat{y} \in B(x'_B, r'_B) \subset B$, and thus we can find a point $\hat{x} \in E_1$ with $|\hat{y} - \hat{x}| < \epsilon_1 C_2 r_B$ (using (2.87)). If W' is the $(n-d)$-dimensional plane which contains S', we put $W = W' + (\hat{x} - \hat{y})$. Hence we let S denote the sphere in W with center $\operatorname{center}(S') + (\hat{x} - \hat{y})$ and radius equal to that of S'. We claim that S satisfies (2.84), (2.85) and (2.86) relative to the pair (x_B, r_B). Note first that

$$S \subset N_{2C_2\epsilon_1 r_B}(S'). \tag{2.88}$$

We show that $\operatorname{dist}(E_1, N_{2C_2\epsilon_1 r_B}(S')) > \epsilon_0 r_B$. Let $s' \in S'$ and $y \in E_1$ be closest to each other. Since $s' \in B$, we must have $y \in 2B$. Let $s \in S$ be closest to s', so $|s - s'| < \epsilon_1 C_2 r_B$. Let $y' \in E_2$ be closest to y; then as $y \in 2B$, $|y - y'| < \epsilon_1 C_2 r_B$; then we have that

$$\begin{aligned}
\operatorname{dist}(E_1, N_{2C_2\epsilon_1 r_B}(S')) &= |y - s'| \geq |y' - s| - |s' - s| - |y - y'| \\
&\geq \operatorname{dist}(E_2, S) - 2\epsilon_1 C_2 r_B \\
&\geq c_1 \epsilon_0 r'_B - 2\epsilon_1 C_2 r_B \\
&\geq \frac{c_1}{C_1}\epsilon_0 r_B - 2\epsilon_1 C_2 r_B.
\end{aligned}$$

Now, choosing ϵ_1 small enough (depending on ϵ_0) and c_1 sufficiently large (depending on C_1), it follows that

$$\operatorname{dist}(E_1, S) \geq \operatorname{dist}(E_1, N_{2C_2\epsilon_1 r_B}(S')) > \epsilon_0 r_B.$$

This proves (2.84) for (x_B, r_B).

We now need to show that we can contract S to a point inside the set

$$\{y \in B(x_B, r_B) \mid \operatorname{dist}(y, E_1) > \epsilon_0 r_B\}\,.$$

To see this, we use (2.88): if we denote by Q_t the contraction of S' to a point, then $\operatorname{dist}(Q_t, E_2) > c_1\epsilon_0 r_B$. Denote by $\{T_t\}_{0 \leq t \leq 1}$ the homotopy $T_t(x) = x + t(\hat{y} - \hat{x})$, so that $T_0(S) = S$, $T_1(S) = S'$ and $T_t(S')$ is a $(n-d-1)$-dimensional sphere lying in the $\operatorname{ch}(S \cup S')$. Then we see that $T_t(S) \subset N_{2C_2\epsilon_1 r_B}(S')$, so $\operatorname{dist}(T_t(S), E_1) \geq \epsilon_0 r_B$. Thus, putting

$$\widetilde{T}_t(x) := \begin{cases} T_{2t}(x) & \text{for } 0 \leq t \leq \frac{1}{2} \\ Q_{2t-1} & \text{for } \frac{1}{2} \leq t \leq 1, \end{cases}$$

we see that $\widetilde{T}_t$ is the desired contraction; this settles (2.85). Moreover, (2.86) holds from the definition of S. But this implies that (x_B, r_B) belongs to $\mathscr{B}_{E_1}^{GWEC}(\epsilon_0)$. This is impossible, and so no sphere S' satisfying (2.84) to (2.86) can exists, and therefore $(x'_B, r'_B) \in \mathscr{G}_{E_2}^{GWEC}(c_1\epsilon_0)$ for c_1 appropriately chosen (depending on C_1), and ϵ_1 sufficiently small. □

We can now apply Lemma 2.25 (and use the fact that $\text{GWEC}(Q_0, \epsilon) \lesssim \text{BWGL}(Q_0, c\epsilon) \lesssim \beta_E(Q_0)$ for some $c > 0$), to obtain the following corollary.

Corollary 2.34. *Let E be lower content regular, let $Q_0 \in \mathcal{D}$. Then*

$$\beta(Q_0)^2 \sim \mathcal{H}^d(Q_0) + \text{GWEC}(Q_0, \epsilon).$$

8. Appendix to Chapter 2

In this section we prove Theorem 2.1. We begin by recalling the original Traveling Salesman Theorem for higher dimensional sets from [**AS18**, Theorem 3.2 and 3.3].

Theorem 2.35. *Let $1 \leq d < n$ and $E \subseteq \mathbb{R}^n$ be a closed. Suppose that E is lower content (d, c_1)-regular and let $\mathcal{D}$ denote the Christ-David cubes for E.*

(1) Let $\mathsf{C}_0 > 1$ and $\mathsf{A} > \max\{\mathsf{C}_0, 10^5\} > 1$, $p \geq 1$, and $\epsilon > 0$ be given. For $R \in \mathcal{D}$, let Then for $R \in \mathcal{D}$,

$$\mathcal{H}^d(R) + \text{BLWG}(R, \epsilon, \mathsf{C}_0) \lesssim_{\mathsf{A},n,c,\mathsf{C}_0\epsilon} \ell(R)^d + \sum_{Q \subseteq R} \beta_E^{d,p}(\mathsf{A}B_Q)^2 \ell(Q)^d. \tag{2.89}$$

Furthermore, if the right hand side of (2.89) is finite, then E is d-rectifiable.

(2) For any $\mathsf{A} > 1$ and $1 \leq p < p(d)$, there is $\mathsf{C}_0 \gg \mathsf{A}$ and $\epsilon_0 = \epsilon_0(n, \mathsf{A}, p, c) > 0$ such that the following holds. Let $0 < \epsilon < \epsilon_0$. Then

$$\ell(R)^d + \sum_{Q \subseteq R} \beta_E^{d,p}(\mathsf{A}B_Q)^2 \ell(Q)^d \sim_{\mathsf{A},n,c,\epsilon} \mathcal{H}^d(R) + \text{BLWG}(R, \epsilon, \mathsf{C}_0) \tag{2.90}$$

If we set

$$\beta_{E,\mathsf{A},p}(R) := \ell(R)^d + \sum_{Q \subseteq R} \beta_E^{d,p}(\mathsf{A}B_Q)^2 \ell(Q)^d,$$

we will now show

$$\beta_{E,\mathsf{A},p}(R) \sim_{\mathsf{A},p} \beta_{E,3,2}(R) =: \beta_E(R). \tag{2.91}$$

Indeed, one can check that $\beta_E^{d,p}(3B_Q) \lesssim_{\mathsf{A},d,p} \beta_E^{d,p}(\mathsf{A}B_Q)^2$ [**AS18**, Lemma 2.11]. Moreover, note that for every $Q \subseteq R$, if Q^N denotes the Nth ancestor of Q, then there is N so that $3B_{Q^N} \supseteq AB_Q$. With these observations, we have

$$\beta_{E,3,p}(R) \lesssim_{\mathsf{A},p} \beta_{E,\mathsf{A},p}(R) \lesssim_N \ell(R)^d + \sum_{Q^N \subseteq R} \beta_E^{d,p}(\mathsf{A}B_Q)^2 \ell(Q)^d \lesssim_p \beta_{E,3,p}(R).$$

Furthermore, by [**AS18**, Lemma 2.13], we see that $\beta_E^{d,1} \lesssim \beta_E^{d,p}$ for all $p > 1$. Thus, by the Traveling Salesman Theorem, for $A \gg \mathsf{C}_0 \gg 3$

$$\beta_{E,3,p}(R) \lesssim \text{BLWG}(R, \epsilon, \mathsf{C}_0) \lesssim \beta_{E,\mathsf{A},1}(R) \lesssim \beta_{E,3,1}(R) \lesssim \beta_{E,3,2}(R).$$

This completes the proof of Theorem 2.1.

CHAPTER 3

Sets with topology

1. Introduction

In the previous chapter we settled the question raised in (3.1), about whether one could prove 'families' of Analyst's travelling salesman theorem corresponding to the various quantitative properties the David and Semmes introduced to characterise rectifiability. In this chapter, we will address the issue, which was contextualized in Chapter 1, (2.2), of finding a geometric object which could function as curve in higher dimensions, at least from the point of view of the TST. However, the type of problems that we will deal with here can be seen from different angles, and find different motivations.

1.1. Topological non-degeneracy of sets and geometric complexity. In [S], Semmes stated the following guiding principle to understand the relation between the topology of some set, and its 'mass' distribution.

'Suitable topological conditions on a space in combination with upper bounds on the mass often implies serious restrictions on the geometric complexity of the space.'

In the monograph [DS00], David and Semmes made this principle into the following theorem — to avoid introducing extra notation, we state it in a somewhat imprecise manner.

Theorem 3.1 ([DS00], Theorem 0.10). *Let E be a compact subset of $\mathbb{R}^n$ and let A be a union of dyadic cubes in $\mathbb{R}^n$ containing E. If $\mathcal{H}^d(E) < +\infty$ and if, given a constant $\theta > 0$, $\mathcal{H}^d(f(E)) > \theta$ for every continuous mapping $f : E \to A$ which is homotopic to the identity through mappings from E to A, then the following holds. For any $\tau > 0$, there is a compact set $Z \subset \mathbb{R}^n$ such that*

- *Z is Ahlfors regular, uniformly rectifiable and contains big pieces of Lipschitz graphs;*
- *$\mathcal{H}^d(Z) > \theta'$, where θ' depends on n and A, but not on τ.*
- *$\mathcal{H}^d(Z \setminus E) \leq \tau \mathcal{H}^d(E)$.*

An immediate consequence of the main result of this chapter is Theorem 3.8; this is similar to Theorem 3.1, but starts from somewhat different initial assumptions: in particular, we will be looking at deformations which are only Lipschitz. Let us remark the following: the author's scarce knowledge on this matter means that he cannot tell whether Theorem 3.8 contains enough difference to Theorem 3.1 to deserve mention.

1.2. Uniformly non-flat sets and their Hausdorff dimension. A third question which motivates the results of this chapter stems from a result of Bishop and Jones, [BJ97]. Here, they proved that if a connected compact subset of the plane is uniformly non-flat, then its dimension is strictly larger than one. For definitions and the precise statement of this, see Section 11. Moreover, their theorem showed explicitly how the non-flatness of the set affects the lower bound on its dimension. In [Dav04], David gave a corresponding result for higher dimensional sets, which however is qualitative in nature, i.e. it does not present how the non-flatness of E affect the dimensionality of E. Here, as a further application of our main result, we give an exact analogue of Bishop and Jones's result.

1.3. The topological condition on E. Let us now define precisely the topological condition mentioned in Chapter 1, (3.2). Let E be a *closed* subset of $\mathbb{R}^n$.

Definition 3.2 (Allowed Lipschitz deformations with parameter α_0). Fix a constant $0 < \alpha_0 < 1$. Consider a one parameter family of Lipschitz maps $\{\varphi_t\}$, $0 \leq t \leq 1$, defined on $\mathbb{R}^n$. We say that $\{\varphi_t\}_{0 \leq t \leq 1}$ is an *allowed Lipschitz deformation* with parameter α_0, or an α_0-ALD, if it satisfies the following four conditions:

$$\varphi_t(B(x,r)) \subset \bar{B}(x,r) \text{ for each } t \in [0,1]; \tag{3.1}$$

$$\text{for each } y \in \mathbb{R}^n, t \mapsto \varphi_t(y) \text{ is a continuous function on } [0,1]; \tag{3.2}$$

$$\varphi_0(y) = y \text{ and } \varphi_t(y) = y \text{ for } t \in [0,1] \text{ whenever } y \in \mathbb{R}^n \setminus B(x,r); \tag{3.3}$$

$$\operatorname{dist}(\varphi_t(y), E) \leq \alpha_0 r \text{ for } t \in [0,1] \text{ and } y \in E \cap B(x,r), \text{ where } 0 < \alpha_0 < 1. \tag{3.4}$$

The topological condition that we impose on E is the following.

Definition 3.3 (Topological Condition). Fix four parameters:

$$r_0, \text{ the scale parameter,} \tag{3.5}$$

$$\alpha_0, \text{ the distance parameter,} \tag{3.6}$$

$$\delta_0, \text{ the lower regularity parameter,} \tag{3.7}$$

$$\eta_0, \text{ the boundary parameter.} \tag{3.8}$$

We say that a subset $E \subset \mathbb{R}^n$ satisfies the topological condition with parameters r_0, α_0, δ_0 and η_0, or the $(r_0, \alpha_0, \delta_0, \eta_0)$-(TC), or just (TC), if for all α_0-ALD $\{\varphi_t\}$, and for all $x_0 \in E$ and $0 < r < r_0$, we have

$$\mathcal{H}^d\left(B(x_0, (1-\eta_0)r) \cap \varphi_1(E)\right) \geq \delta_0 r^d. \tag{TC}$$

We may refer to a set E satisfying the topological condition above as a *topologically stable* d-surface, or, for short, TS d-surface.

Remark 3.4. Let us remark once more that this condition is not new. As stated it was introduced by G. David in [**Dav04**], where he proved that a set E endowed with such a condition and so that its β numbers are large, must have dimension strictly larger than d.

1.4. Statement of the main result and some consequences. Let $\mathcal{D}(E) = \mathcal{D}$ denote the family of Christ-David cubes relative to E (see Theorem 5.2 for definitions). Our main result in the following.

Theorem 3.5. *Let E be a closed subset of $\mathbb{R}^n$ and let $0 < \eta_0, \delta_0, \alpha_0 < 1$ and $r_0 > 0$. If E satisfies the $(r_0, \alpha_0, \delta_0, \eta_0)$-(TC), then, for any $Q_0 \in \mathcal{D}$ such that*

$$\ell(Q_0) < r_0, \tag{3.9}$$

and any $\mathsf{A} \geq 1$,

$$\operatorname{diam}(Q_0)^d + \sum_{\substack{Q \in \mathcal{D} \\ Q \subset Q_0}} \beta_E^{p,d}(\mathsf{A}Q)^2 \ell(Q)^d \leq C\, \mathcal{H}^d(Q_0), \tag{3.10}$$

where the constant C depends on A, α_0, δ_0 and η_0.

Remark 3.6. The assumption (3.9) is a natural one and cannot be avoided. Assuming the topological condition from a certain scale, i.e. from r_0, means that at larger scale there could be holes. This would make the term $\mathrm{BWGL}(Q_0)$ come back.

The assumption that E is closed is relevant, but not restrictive; in fact we have that $\mathcal{H}^d(E) = \mathcal{H}^d(\overline{E})$.

Theorem 3.5 together with the main result from [**AS18**], see Theorem 2.1 in Chapter 2, gives the following corollary.

Corollary 3.7. *Let E be a closed subset of $\mathbb{R}^n$. If there are parameters $\eta_0, \delta_0, \alpha_0$ and r_0 so that E satisfies the topological condition (TC), then with $1 \leq p < p(d)$, where*

$$p(d) := \begin{cases} \frac{2d}{d-2} & \text{if } d > 2 \\ \infty & \text{if } d \leq 2 \end{cases}, \tag{3.11}$$

and A sufficiently large (depending only on n), we have

$$\operatorname{diam}(Q_0)^d + \sum_{\substack{Q \in \mathcal{D} \\ Q \subset Q_0}} \beta_E^{p,d}(\mathsf{A}Q)^2 \ell(Q)^d \sim \mathcal{H}^d(Q_0), \tag{3.12}$$

where $Q_0 \in \mathcal{D}$ is so that $\ell(Q_0) < r_0$, and where the constant behind the symbol $\sim$ depends on A, n, p*, the parameters coming from the topological condition, and the parameters behind the constants appearing in Theorem 2.1 of Chapter 2.*

In some cases, the estimate (3.12) was already known: for ϵ-Reifenberg flat (see Definition 3.13), for example. Another known case was for $n = d - 1$ and E is satisfying Condition B (for definitions, see [**AS18**] and the references therein). We will see below that both Reifenberg flatness and Condition B imply the topological condition that we introduced above.

Concerning the type of questions raised in Subsection 1.1, we have the following corollary, which pops out immediately from Theorem 3.5 and the theory of uniformly rectifiable sets (see [**DS93**]).

THEOREM 3.8. *Let E be a topologically stable d-surface. If E is upper Ahlfors d-regular, then E is uniformly rectifiable.*

Finally, concerning the Hausdorff dimension of uniformly non-flat sets (see Section 11 for precise definitions), we have the following theorem.

THEOREM 3.9. *Let $E \subset \mathbb{R}^n$ be a topologically stable d-surface. Let $Q_0 \in \mathcal{D}$ be such that, for any $Q \in \mathcal{D}(Q_0)$, we have that*

$$\beta_E^{p,d}(\mathsf{A}\, Q)^2 > \beta_0 > 0. \tag{3.13}$$

Then

$$\dim(Q_0) > d + c\beta_0^2. \tag{3.14}$$

See Section 11 for a sketch of proof of this.

1.5. Acknowledgement. This chapter is based on the preprint [**V19b**]. I would like to thank Jonas Azzam, my supervisor, for his help and support during the writing of the paper. I would also like to thank PCMI/IAS and the organisers of the graduate school of 2018 on Harmonic Analysis: it was here that I learned many tools used in this paper, as explained by Guy David (whom I also thank for the intuitively clear explanations). We should also mention that a lot of the machinery used in this chapter comes from [**Dav04**].

2. Preliminaries

For $j \in \mathbb{Z}$, we will denote by Δ_j the family of dyadic cubes with side length 2^{-j}. We also set

$$\Delta := \bigcup_{j \in \mathbb{Z}} \Delta_j.$$

For a cube $I \in \Delta$, we write

$$\partial_d I \tag{3.15}$$

to denote *the d-dimensional skeleton* of I. Given a dyadic cube I in $\mathbb{R}^n$, the d-dimensional skeleton of I is just the union of all its d-dimensional faces. We also set

$$\mathcal{S}_{j,d} := \bigcup_{I \in \Delta_j} \partial_d I. \tag{3.16}$$

Let us remark that for a set V, we write ∂I to mean the standard boundary of V; so in particular $\partial I = \partial_{n-1} I$.

REMARK 3.10. In this chapter we will adopt the dyadic system of cubes of Christ-David, as per Theorem 5.2.

2.1. Constants. We will have many constant floating around in this chapter. Let us list them.

(1) c_1: the lower content regularity constant, as defined in (1.14).
(2) A: this is the constant that determines how much we inflate the balls when computing the β numbers, as, for example, in Theorem 3.5
(3) ϵ: Reifenberg-flatness parameter.
(4) c_1: how far the sphere has to be from the set E in the definition of Semmes surfaces.

(5) C_1: expansion factor of top cubes in Lemma 2.4.
(6) τ: smoothing parameter in Lemma 2.4.
(7) k_0: generation parameter in Lemma 2.4.
(8) M: constant for the stopping time in the construction of Lemma 2.4.
(9) λ: nets parameter in Theorem 5.2.
(10) c_5: containment parameter in Theorem 5.2.
(11) $r_0, \alpha_0, \eta_0, \delta_0$: parameters of the topological condition (TC).
(12) $r_1, \alpha_1, \eta_1, \delta_1$: parameters for the skeletal topological condition (see (3.43)).
(13) C_2: constant of the skeletal topological condition.
(14) C_3: Ahlfors regularity constant of the approximating set E_R (and of E_ρ).
(15) ρ: scale parameter of the approximating set E_ρ (see section 8).
(16) σ: scale parameter for the construction of the domain of the fucntional J.
(17) M: large constant in the functional J (not the same M as above!).
(18) c_2: small constant in the definition of M.
(19) k: quasiminimality: Hausdorff measure constant.
(20) δ: quasiminimality: locality constant.
(21) c_3: small constant in the definition of δ.
(22) C_4: inflation constant for the β numbers on Z_Q.

3. Some remarks on the topological condition

We would like to motivate a little bit our choices: why would one use the topological condition as in Definition 3.3? A quantitative bound as in (3.12) was already known for surfaces satisfying the so called Condition B and for Reifenberg flat sets; as mentioned above, both of them imply the topological condition (TC).

As Condition B applies only to subsets of codimension one, let us consider instead a more general property which make sense in any codimension. Subsets satisfying this property are called Semmes surfaces. They were first introduced by David in [Dav88].

Definition 3.11. Let n, d be two integers with $0 \leq d \leq n-1$. A *Semmes surface* is a subset $E \subset \mathbb{R}^n$ so that the following holds. Let $c_1 < 1$ be a constant. For all points $x_0 \in E$ and radii $r > 0$, we can find and affine subspace W of dimension $n-d$ and a sphere S of dimension $n-d-1$ which is contained in W and so that

$$S \subset B(x_0, r) \tag{3.17}$$

$$\operatorname{dist}(S, E) \geq c_1 r \tag{3.18}$$

$$S \text{ links } E. \tag{3.19}$$

Let us explain what we mean by S links E; we say that S and E are linked if it is not possible to find an homotopy $F(x,t)$ defined and continuous for all $(x,t) \in \mathbb{R}^n \times [0,1]$ such that

$$F(x,t) = x \text{ for } t = 0 \text{ and for } x \in \mathbb{R}^n \setminus B(x_0, 10r); \tag{3.20}$$

$$F(x,1) \in \mathbb{R}^n \setminus B(x_0, 10r) \text{ for all } x \in E; \tag{3.21}$$

$$F(x,t) \in \mathbb{R}^n \setminus S \text{ for all } x \in E \text{ and for all } 0 \leq t \leq 1. \tag{3.22}$$

Note that a set satisfying Condition B is just a d-dimensional Semmes surface with $d = n-1$. David shows the following.

Lemma 3.12 ([Dav04], Lemma 2.16). *A d-dimensional Semmes surface satisfies the topological condition* (TC) *with parameters depending on c_1.*

Let us now turn to Reifenberg flat sets.

Definition 3.13. Let n, d as above, and fix a positive constant $\epsilon > 0$. A subset $E \subset \mathbb{R}^n$ is called a d-dimensional ϵ-*Reifenberg flat* set if for all $(x,r) \in E \times \mathbb{R}_+$, there exists a d-dimensional affine plane P so that

$$d_{x,r}(E, P) < \epsilon, \tag{3.23}$$

where $d_{x,r} = d_{B(x,r)}$ is as in (1.7).

LEMMA 3.14. *Fix $\epsilon > 0$ sufficiently small and let E be a closed subset of $\mathbb{R}^n$. If E is a d-dimensional ϵ-Reifenberg flat set, then there exists constants $\alpha_0, \delta_0, \eta_0$ which depend only on ϵ so that E is a TS d-surface (with these parameters)..*

The lemma follows from Reifenberg topological disk theorem.

4. First reductions and the construction of approximating skeleta

4.1. Lower content regularity. Let us get started with the proof of Theorem 3.5. Fix a top cube $R \in \mathcal{D}$. First, we see that if $\mathcal{H}^d(R) = \infty$, then there is nothing to prove. Thus, we may (and will) assume that

$$\mathcal{H}^d(R) < +\infty \tag{3.24}$$

We can also take E to be compact, since Theorem 3.5 is local.

To obtain the estimates on β numbers that we want, we would like to apply the coronisation by Ahlfors regular sets proved in Chapter 2 (see Main Lemma 2.4). To do so, we first need to show that any topologically stable d-surface is lower content d-regular.

LEMMA 3.15. *Let $E \subset \mathbb{R}^n$ be compact subset which satisfies the topological condition (TC) with parameters r_0, α_0, δ_0 and η_0. Then E satisfies*

$$\mathcal{H}^d_\infty(E \cap B(x,r)) \gtrsim \mathsf{c}_1 r^d$$

for all $x \in E$ and $r < r_0$; the lower regularity constant c_1 will depend on δ_0 and η_0.

This fact is essentially present in Chapter 12 of [DS00], although in a somewhat different form. We give a proof for this reason. We will first prove the following Sublemma, which will imply Lemma 3.15.

SUBLEMMA 3.16. *Let E be a compact subset of $\mathbb{R}^n$ and let $(x,r) \in E \times \mathbb{R}_+$ be a pair so that*

$$\mathcal{H}^d_\infty(B(x,r) \cap E) < \mu\nu\, \delta_0 r^d \tag{3.25}$$

for a parameter ν (sufficiently small depending on η_0) and a number $\mu > 0$ which depend only on η_0 and δ_0 (as in Lemma 3.15). Then there exists a one parameter family of Lipschitz mappings $\{\varphi_t\}$ which satisfies (3.1)-(3.4) and so that φ_1 maps $B(x,(1-\eta_0)r) \cap E$ into the $(d-1)$-dimensional skeleton of cubes from Δ_j, where $j = j(\rho) \in \mathbb{N}$ is such that $2^{-j} \sim \rho$, and $\rho = (\nu\delta_0)^{1/d} r$.

The proof of this Sublemma will follow quickly if we use the following proposition from [DS00].

PROPOSITION 3.17 ([DS00], Proposition 12.61). *Let A be a union of dyadic cubes from Δ_j, where j is some integer. There is a possibly small constant $c > 0$ so that if $\theta \sim c\,2^{-j}$, the following is true. Let F be a compact subset of A such that*

$$\mathcal{H}^d_\infty(F \cap I) < \theta \text{ for all } I \in \Delta_j. \tag{3.26}$$

Then there is a Lipschitz mapping $\phi : F \to A$ so that $\phi(F) \subset \mathcal{S}_{j,d-1}$ and $\phi(F \cap I) \subset I$ for all $I \in \Delta_j$. Also, ϕ is homotopic to the identity through mappings from F to A.

PROOF OF SUBLEMMA 3.16. Let $\rho > 0$ and $j(\rho) \in \mathbb{N}$ be as in the statement of the Sublemma, and let $\mu > 0$, $\nu > 0$ two possibly small parameteres to be fixed soon. Set

$$A^1 := \bigcup \left\{ I \in \Delta_{j(\rho)} \,|\, I \cap B\left(x, (1-\mu)r\right) \neq \emptyset \right\}, \tag{3.27}$$

$$A^2 := \bigcup \left\{ I \in \Delta_{j(\rho)} \,|\, I \cap A^1 \neq \emptyset \right\}. \tag{3.28}$$

We want μ and ν to be so that

$$\eta_0 > 10\,\mu > 30(\nu\,\delta_0)^{1/d}. \tag{3.29}$$

This choice then implies that

$$E \cap B(x,(1-\eta_0)r) \subset E \cap B(x,(1-\mu)r) \subset E \cap A^1 \subset E \cap A^2 \subset B(x,r) \cap E, \tag{3.30}$$

Now, by the hypothesis (3.25), we see that for any $I \in \Delta_{j(\rho)}$ which is also contained in A^2 we have

$$\mathcal{H}^d_\infty\left(I \cap (E \cap A^1)\right) < \mu\,\nu\,\delta_0\, r^d = \mu\,\rho^d.$$

Adjusting the choice of μ and ν if needed, we see that this implies (3.26) to hold for all $I \in \Delta_{j(\rho)}$ which also lie in A^2 with $F = E \cap A^1$. Moreover, with this F, (3.26) holds trivially for any other $I \in \Delta_{j(\rho)}$. Hence we apply Proposition 3.17 with $j = j(\rho)$ (i.e. so that $2^{-j} \sim \rho$), $A = A^2$ as defined in (3.28) and $F = A^1 \cap E$, as defined in (3.27). We obtain a Lipschitz mapping ϕ which sends $E \cap A^1$ into $\mathcal{S}_{j(\rho),d-1}$ and all the properties listed in the proposition. Note in particular that with the choice (3.29) of μ and ν and the fact that $\phi(E \cap I) \subset I$ for any $I \in \Delta_{j(\rho)}$, we have that

$$B\left(x,(1-\eta_0)r\right) \cap \phi(E) = B\left(x,(1-\eta_0)r\right) \cap \phi(E \cap A^1) \subset \phi(E \cap A^1). \tag{3.31}$$

Now we can extend ϕ to be the identity outside of A^2. Setting

$$\varphi_t(y) = t\phi(y) + (1-t)y \text{ for } t \in [0,1], \tag{3.32}$$

it is easy to check that φ_t satisfies (3.1)-(3.4). □

Proof of Lemma 3.15. For the sake of contradiction, suppose that for $x \in E$, and $r < r_0$, the inequality (3.25) holds. Then, using the definition of topological condition (TC) (which can be applied since $r < r_0$), we obtain

$$\begin{aligned} \delta_0 r^d &< \mathcal{H}^d\left(B(x,(1-\eta_0)r) \cap \varphi_1(E)\right) \\ &= \mathcal{H}^d\left(B(x,(1-\eta_0)r) \cap \phi(E)\right) \\ &\overset{(3.31)}{\leq} \mathcal{H}^d\left(\phi(E \cap A^1)\right) = 0. \end{aligned}$$

Thus we must have that for any such a pair $(x,r) \in E \times \mathbb{R}_+$, (3.25) cannot hold. This implies the lower content d-regularity of E (for scales smaller than r_0), with constant c_1 depending only on δ_0 and η_0. □

Remark 3.18. Because all our statements are local, we will be ignoring the fact that our set is lower regular only for (possibly) small scales. In fact, we could assume without loss of generality that $r_0 = 1$.

Remark 3.19. Becuase E is lower content regular, we can apply Theorem 2.1; in particular, as it was shown in the Appendix to Chapter 2 (see (2.91)), for any cube $R \in \mathcal{D}$ with $\ell(R) \leq 1$, we have

$$\beta_{E,3,2}(R) \approx_{\mathsf{A},n,p,\mathsf{c}_0,\mathsf{C}_0} \beta_{E,\mathsf{A},p}(R).$$

It sufficies therefore to prove Theorem 3.5 with $p = 2$ and $\mathsf{A} = 3$.

4.2. Construction of the approximating skeleta E_R. In this subsection we recall the corona construction from Chapter 2. Recall that this is a corona decomposition of a lower regular set E (in the sense of (1.14)) in terms of Ahlfors regular sets E_R (as in (1.2)). See Main Lemma 2.4. We can apply Main Lemma 2.4 because of Lemma 3.15.

For readability purposes, we give a short sketch of the proof of Lemma 2.4; it will be just a brief summary of the material from Chapter 2, Section 3, but it will help us to set some notation which will be needed in this chapter, too.

Recall that the proof of Lemma 2.4 started off with a Frostmann's Lemma type argument. Assume without loss of generality that $Q_0 \subset [0,1]^n$; let us fix some notation:

$$\begin{aligned} \Delta_j(Q_0) &:= \{I \in \Delta_j \,|\, Q_0 \cap I \neq \emptyset\}\,; \\ \Delta(Q_0) &:= \bigcup_{j \geq 0} \Delta_j(Q_0). \end{aligned}$$

We also set

$$V_j(Q_0) := \bigcup_{I \in \Delta_j(Q_0)} I.$$

Next, we iteratively defined a measure on the approximating set V_j; we then put all those dyadic cubes where this measure is too large in a family called Bad. Recall that we have the packing condition

$$\sum_{I \in \mathsf{Bad}(m)} \ell(I)^d \leq C(n,d), \mathcal{H}^d(Q_0), \tag{3.33}$$

which is independent of $m \in \mathbb{N}$. This is proven in Chapter 2, (2.17).

Let now $k_0 > 0$ be an arbitrary integer number, $M > 1$ a constant to be fixed later and $\mathsf{A} > 1$ the inflation constant for the β numbers (see Constant (2)). Recall the stopping time that we did in Chapter 2: we start with putting $Q_0 \in \mathsf{Tree}(Q_0)$. If there exists a cube $Q \in \mathsf{Child}(Q_0)$ such that

$$MB_Q \cap I \neq \emptyset \qquad \text{and}$$
$$\lambda \ell(I) \leq \ell(Q) \leq \ell(I), \tag{3.34}$$

where λ is as in Theorem 5.2 and B_Q as in (5.4), then we stop. Otherwise we put all the cubes $Q \in \mathsf{Child}(Q_0)$ in $\mathsf{Tree}(Q_0)$; next we scan $\mathsf{Child}(Q_0)$ to see if any cube here has a child satisfying (3.34). We proceed recursively in this fashion; the process will eventually terminate because we stopped at all cubes, or because we reached the bottom of $\mathcal{D}(k_0)$. Furthermore, we consider all cubes Q of the same generation of Q_0, so that

$$2\mathsf{A}\, Q_0 \cap Q \neq \emptyset,$$

where A is the constant appearing in (2). We denote this family by $\mathcal{N}(Q_0)$. On each of these cubes, we perform the same stopping time, so to construct the relative $\mathsf{Tree}(Q)$. Finally we put

$$\mathsf{Forest}(Q_0) := \bigcup_{Q \in \mathcal{N}(Q_0)} \mathsf{Tree}(Q), \tag{3.35}$$

and also

$$\mathsf{Stop}(Q_0) := \{Q \in \mathcal{D}(k_0) \,|\, Q \text{ is minimal in } \mathsf{Forest}(Q_0)\}.$$

Next, we put

$$\mathsf{Next}(Q_0) := \bigcup_{Q \in \mathsf{Stop}(Q_0)} \mathsf{Child}(Q).$$

We now repeat the stopping time on each $R \in \mathsf{N}ext(Q_0)$. Thus, if we set $\mathsf{Top}_0(k_0) := \{Q_0\}$, then $\mathsf{Top}_1(k_0) := \mathsf{Next}(Q_0)$; proceeding inductively, supposed that $\mathsf{Top}_m(k_0)$ has been defined, for $m \in \mathbb{N}$: we put

$$\mathsf{Top}_{m+1}(k_0) := \bigcup_{R \in \mathsf{Top}_m(k_0)} \mathsf{Next}(R).$$

Finally, we set

$$\mathsf{Top}(k_0) = \bigcup_{k=0}^{\infty} \mathsf{Top}_k \tag{3.36}$$

Hence, to each element $R \in \mathsf{Top}$, there correspond a forest $\mathsf{Forest}(R)$ and a family of minimal cubes $\mathsf{Stop}(R)$. Now, for each $R \in \mathsf{Top}$, let us define

$$d_R(x) := \inf_{Q \in \mathsf{Stop}(R)} \left(\ell(Q) + \operatorname{dist}(x, Q)\right), \text{ and} \tag{3.37}$$

$$d_R(I) := \inf_{x \in I} d_R(x), \text{ whenever } I \in \Delta. \tag{3.38}$$

This is a now standard smoothing procedure which goes back to David and Semmes' [**DS91**]. Hence, for a parameter $\tau > 0$, we put

$$\mathscr{C}_R := \{ \text{ maximal } I \in \Delta \,|\, I \cap 2\mathsf{A}\, R \neq \emptyset \text{ and } \ell(I) < \tau d_R(I)\}. \tag{3.39}$$

Finally we set

$$\widetilde{E}_R := \bigcup_{I \in \mathscr{C}_R} \partial_d I.$$

Thus $\widetilde{E}_R$ is the union of d-dimensional skeleta (see (3.15)) of cubes belonging to $\widetilde{E}_R$. Recall from Lemma 2.17 that $\widetilde{E}$ is Ahlfors d-regular (with some constant c_0).

The following lemmas summarise some of the properties of the cubes in $\mathscr{C}_R$.

Lemma 3.20. *The cubes $I \in \mathscr{C}_R$ have disjoint interior and satisfy the following properties.*

1. *If $x \in 15\, I$, for some $I \in \mathscr{C}_R$, then $\ell(I) \approx \tau d(x)$.*
2. *There is a constant depending on τ, such that if $15I \cap 15J \neq \emptyset$ for $I, J \in \mathscr{C}_R$, then $\ell(I) \sim_\tau \ell(J)$.*

Property (1) is just Lemma 2.11, while property (2) is Lemma 2.12.

LEMMA 3.21. *Let S be a cube in $\mathsf{Stop}(Q)$ for some $Q \in \mathsf{Next}(R)$, $R \in \mathsf{Top}(k_0)$. Then there exists a dyadic cube $I_S := I \in \mathscr{C}_Q$ so that $I_S \subset \frac{1}{2}B_S$ and $\ell(I_S) \sim \tau\ell(S)$.*

LEMMA 3.22. *Let $I \in \mathscr{C}_Q$ for $Q \in \mathsf{Next}(R)$, $R \in \mathsf{Top}(k_0)$. Then there exists a cube $Q_I \in \mathsf{Tree}(Q)$ so that*

$$\ell(I) \leq \ell(Q_I) \leq c\tau^{-1}\ell(I);$$
$$\mathrm{dist}(I, Q_I) \leq c\tau^{-1}\ell(I).$$

The proof of lemmas 3.21 and 3.22 are standard. For convenience of the reader, we include them in the Appendix to Chapter 3.

4.3. Modification of $\widetilde{E}_R$. In this subsection we modify slightly the construction of $\widetilde{E}_R$; we need to do so to construct a coherent Federer-Fleming projection in the next section.

Fix $R \in \mathsf{Top}$; recall the definition of $\mathscr{C}_R$ in (3.39). Take a cube $I \in \mathscr{C}_R$. Consider one of its $(n-1)$-dimensional faces, and denote it by T_I. Set

$$\begin{aligned}&\mathrm{Adj}^{n-1}(T_I)\\ &:= \{J \in \mathscr{C}_R \,|\, \ell(J) \leq \ell(I),\ J \cap T_I \text{ is an } (n-1)\text{-face of } J \text{ and } J \cap T_I \subset \mathrm{Int}(T_I)\}\end{aligned}$$

We order the cubes in $\mathscr{C}_R$ from the largest to the smallest one, and we label them as $I_0, ..., I_N$, for some $N \in \mathbb{N}$. This is true because the cardinality of $\mathscr{C}_R$ is finite (depending on k_0). Let us start our construction with $I_0 \in \mathscr{C}_R$ (thus I_0 is the largest cube in $\mathscr{C}_R$). We look at one of its $(n-1)$-dimensional faces, let us denote it by T_{I_0}. Now, let I be a cube of minimal side length contained in $\mathrm{Adj}^{n-1}(T_{I_0})$; let $n(I) \in \mathbb{N}$ be such that $\ell(I) = 2^{-n(I)}$. We consider the family of cubes in $\Delta_{n(I)}$ such that they have an $(n-1)$-dimensional face contained in T_{I_0}. We call this family $\Delta_{n(I)}(T_{I_0})$. Let us denote by

$$\mathscr{D}^{n-1}(T_{I_0}) \tag{3.40}$$

the family of $(n-1)$-dimensional faces of the same side length of I, such that they are both an $(n-1)$-dimensional face of a cube $J \in \Delta_{n(I)}(T_{I_0})$ and also they are contained in T_{I_0}. We may refer to this family as the *tiles* of T_{I_0}. We repeat the same procedure for $I_1, ..., I_N$; we don't do anything if $\mathrm{Adj}^{n-1}(T_{I_j}) = \emptyset$ for some face T_{I_j} of I_j, $1 \leq j \leq N$. Note that the definition of $\mathrm{Adj}^{n-1}(T_I)$ imposes the following: if two cubes I and I' are so that, say, $\ell(I) > \ell(I')$ and $I' \in \mathrm{Adj}^{n-1}(T_I)$, then the tiles constructed on T_I will be the same one that we have on the face $T_{I'} \subset T_I$. The construction of tiles on the other $(n-1)$-faces of I' will not change the ones already present in $T_{I'}$. This procedure terminates since $\mathscr{C}_R$ is finite.

Once we constructed $(n-1)$-dimensional tiles on all the $(n-1)$-dimensional faces of all cubes in $\mathscr{C}_R$, we rest. After, we proceed as follows. Denote by

$$\mathscr{F}^{n-1} \tag{3.41}$$

the family of $(n-1)$-dimensional faces belonging to some cube in $\mathscr{C}_R$. If $T \in \mathscr{F}^{n-1}$ and $\mathscr{D}^{n-1}(T) \neq \emptyset$, the put the elements of $\mathscr{D}^{n-1}(T)$ in $\mathscr{F}^{n-1}$ and take T away. If $\mathscr{D}^{n-1}(T) = \emptyset$, then leave T in $\mathscr{F}^{n-1}$.

Next, we repeat the previous construction: order the elements of $\mathscr{F}^{n-1}$ in decreasing order with respect to side length and consider T_0 (the largest face in $\mathscr{F}^{n-1}$). For each $(n-2)$-dimensional face F_{T_0} of T_0 we set

$$\begin{aligned}&\mathrm{Adj}^{n-2}(F_{T_0})\\ &:= \left\{T \in \mathscr{F}^{n-1} \,|\, \ell(T) < \ell(T_0),\ T \cap F_{T_0} \text{ is an } (n-2) - \text{face of } T \text{ and } T \cap F_{T_0} \subset \mathrm{Int}(F_{T_0})\right\}.\end{aligned}$$

We now look for the minimal element of $\mathrm{Adj}^{n-2}(F_{T_0})$, and call it T. Let $n(T) \in \mathbb{Z}$ so that $\ell(T) = 2^{n(T)}$; we now tessellate F_{T_0} with tiles of side length $2^{n(T)}$; by tessellate here we mean the obvious thing, i.e. we substitute F_{T_0} with its children of size $2^{n(T)}$. Let us denote the tiles so constructed by

$$\mathscr{D}^{n-2}(F_{T_0}).$$

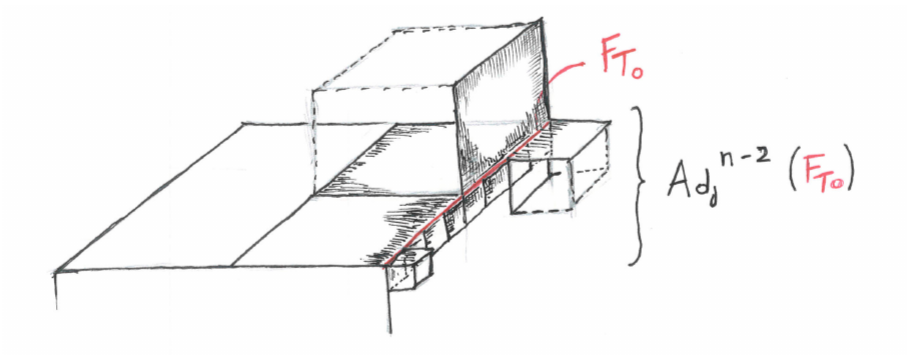

FIGURE 1. The construction of the family of $(n-2)$ dimensional faces $\mathrm{Adj}^{n-2}(F_{T_0})$.

We repeat the same procedure for $T_1, ..., T_{N'} \in \mathscr{F}^{n-1}$. Again, the construction of $(n-2)$-dimensional tiles for smaller $(n-2)$-dimensional faces does not affect the previously constructed tiles for larger faces. This procedure terminates since $\mathscr{F}^{n-1}$ is finite, which follows trivially from $\mathscr{C}_R$ being finite. Next, we set

$$\mathscr{F}^{n-2}$$

to be the family of $(n-2)$-dimensional faces coming from elements of $\mathscr{F}^{n-1}$, and we immediately modify it as above: if $\mathscr{D}^{n-2}(F_T) \neq \emptyset$, for $T \in \mathscr{F}^{n-1}$, we substitute F_T with the corresponding family of tiles.

We continue this construction: we obtain $\mathscr{F}^{n-3}$ from $\mathscr{F}^{n-2}$, $\mathscr{F}^{n-4}$ from $\mathscr{F}^{n-3}$, and so on, until we construct $\mathscr{F}^d$. We stop at this point and we set

$$E_R := \widetilde{E}_R \cup \left(\bigcup_{F \in \mathscr{F}^d} F \right). \tag{3.42}$$

LEMMA 3.23. *The set E_R is Ahlfors d-regular.*

PROOF. Lower regularity follows immediately from the definition and the lower regularity of $\widetilde{E}_R$. On the other hand, note that for any cube $I \in \mathscr{C}_R$, any smaller neighbouring cube $I' \in \mathscr{C}_R$ will satisfy $\ell(I') > \tau\ell(I)$ (using Lemma 3.20, (2)). If we envelope I in cubes of side length $\ell(I)\tau$ and we consider the d-dimensional skeleton of this family of cubes, we see that the overall additional mass will not exceed a constant times $\ell(I)^d$, where such a constant depends on n, d and τ. Thus upper regularity is also preserved. □

NOTATION 3.24. *From now on, we fix the notation for the regularity constant of E_R: it will be denoted by C_3 and depends on n, d, τ anf the regularity constant of $\widetilde{E}_R$.*

5. A topological condition on approximating skeleta

We now introduce a condition on E_R which will imply the existence of a uniformly rectifiable sets lying close to it. This is basically the condition that David calls TND (topological nondegeneracy condition) in [Dav04] with a few changes to adapt it to our trees. Let $R \in \mathsf{Top}$ and E_R be the set constructed in Section 4.2, i.e. the set given in (3.42).

DEFINITION 3.25 (STC). Let C_2 be an arbitrary big constant and let $k_0 \in \mathbb{N}$ be as in the statement of Main Lemma 2.4 (Chapter 2). Then we say that the family of subsets $\{E_R\}_{R \in \mathsf{Top}(k_0)}$ satisfies the *skeletal topological condition* with parameter C_2, or C_2-(STC), if we can find four constants

$$0 < \alpha_1,\ \eta_1,\ \delta_1 < 1 \text{ and } r_1 > 0 \tag{3.43}$$

such that

$$\text{for all } x_1 \in E, \tag{3.44}$$

$$\text{for all } R \in \mathsf{Top}(k_0) \text{ s.t. } x_1 \in R \text{ and } \frac{\ell(R)}{4} \le r_1, \tag{3.45}$$

$$\text{for all } Q \in \mathsf{Tree}(R) \text{ s.t. } x_1 \in Q, \tag{3.46}$$

for which

$$\mathcal{H}^d\left(E_R \cap B(x_1, \ell(Q))\right) \le C_2 \ell(Q)^d \tag{3.47}$$

holds, there is a ball $B(x_2, r_2)$ centered on E and contained in $B(x_1, \ell(Q))$ such that, for each one-parameter family $\{\varphi_t\}_{0\le t\le 1}$ of Lipschitz functions on $\mathbb{R}^n$ that satisfy (3.1), (3.2), (3.3) and

$$\operatorname{dist}(\varphi_t(y), E) \le \alpha_1 \ell(Q) \text{ for } t \in [0,1] \text{ and } y \in E_R \cap B(x_2, r_2), \tag{3.48}$$

we have that

$$\mathcal{H}^d\left(\varphi_1(E_R \cap B(x_2, r_2))\right) \ge \delta_1 \ell(Q)^d + \mathcal{H}^d(E_R \cap A_{\eta_1 \ell(Q)}(x_2, r_2)), \tag{3.49}$$

where

$$A_{\eta_1\ell(Q)}(x_2, r_2) := B(x_2, r_2) \setminus B(x_2, r_2 - \eta_1 \ell(Q)). \tag{3.50}$$

Remark 3.26. Note that

$$r_2 > \eta_1 \ell(Q); \tag{3.51}$$

if $r_2 \le \eta_1 \ell(Q)$, then $A_{\eta_1 \ell(Q)} = B(x_2, r_2)$. Thus if we apply (3.49) with $\varphi_t(y) = y$, then we would obtain that $\mathcal{H}^d(E_R \cap B(x_2, r_2)) > \mathcal{H}^d(E_R \cap B(x_2, r_2))$, a contradiction.

6. Federer-Fleming projections

In this section we will construct a Federer-Fleming projection of E onto a subset of E_R; we will use these projections in the next section to prove that the topological condition (TC) on E implies the condition STC on the approximating skeleta. Our construction will mimic the one in [**Dav04**], which in turn comes from [**DS00**]. The difference here is that we are dealing with a skeleton of faces coming from cubes of different sizes.

Let $B(x, r)$ be a ball centered on the set E (the construction below will be applied to the ball $B(x_2, r_2)$ as in the definition of STC, Definition 3.25). Set

$$\mathscr{C}_R(x, r) := \{I \in \mathscr{C}_R \,|\, I \cap B(x, r) \neq \emptyset\}; \tag{3.52}$$

$$\mathscr{F}^m(x, r) := \{T \in \mathscr{F}^m \,|\, T \cap B(x, r) \neq \emptyset\} \text{ for } d \le m \le n-1; \tag{3.53}$$

$$D_R(x, r) := \bigcup_{I \in \mathscr{C}_R(x, r)} I. \tag{3.54}$$

Furthermore, we set $\mathscr{C}_R^2(x, r)$ to be the family of dyadic cubes composed by $\mathscr{C}_R(x, r)$ together with a maximal subfamily of cubes $J \in \Delta \setminus \mathscr{C}_R$ so that

$$\operatorname{Int}(J) \subset \left(\bigcup_{I \in \mathscr{C}_R(x, r)} I \right)^c; \tag{3.55}$$

$$\text{there exists a dyadic cube } I \in \mathscr{C}_R(x, r) \text{ s.t. } I \cap J \neq \emptyset, \tag{3.56}$$

and moreover, if we let

$\mathcal{N}(J)$ be the family of cubes in $\mathscr{C}_R(x, r)$ which intersect J,

we ask that

$$\min\{\ell(I) \,|\, I \in \mathcal{N}(J)\} \le \ell(J) \le \max\{\ell(I) \,|\, I \in \mathcal{N}(J)\}. \tag{3.57}$$

The family $\mathscr{C}_R^2(x, r) \setminus \mathscr{C}_R(x, r)$ forms a sheath for $\mathscr{C}_R(x, r)$ (imagine the plastic covering of some Minecraft electrical wires). Finally we define

$$D_R^2(x, r) := \bigcup_{I \in \mathscr{C}_R^2(x, r)} I. \tag{3.58}$$

Recall the definition of E_R as in (3.42). The following lemma is similar to Proposition 3.1 in [DS00], and so is the proof. The only difference is that we are working with a non-uniform grid of cubes.

LEMMA 3.27. *Given* $(x, r) \in E \times \mathbb{R}_+$, *there exists a Lipschitz map* $\pi : \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$\pi(y) = y \text{ whenever } x \in \mathbb{R}^n \setminus D_R^2(x, r); \tag{3.59}$$

$$\pi(I) \subset I \text{ if } I \in \mathscr{C}_R^2(x, r); \tag{3.60}$$

$$\pi(E) \cap I \subset E_R \cap I \text{ for any } I \in \mathscr{C}_R(x, r). \tag{3.61}$$

We will obtain our Federer-Fleming projection as the composition of a finite number of maps which we will define inductively. We start by defining a map, let us call it π_1, that will send points in $D_R^2(x, r) \cap E$ into $(n-1)$-dimensional faces. We define π_1 on each individual cube $I \in \mathscr{C}_R^2(x, r)$ as follows. Pick a point $c_I \in I$ such that $c_I \notin E$. This is possible since $\mathcal{H}^d(E) < \infty$ (recall (3.24)) and thus, in particular, $\dim_H(E) < d+1$; a standard argument then shows that E is porous, and thus such a point c_I must exist. Then for $y \in E \cap \mathrm{Int}(I)$, we set

$$\pi_1(y) \text{ to be the point where the line passing through } y \text{ and } c_I \text{ meets } \partial I; \tag{3.62}$$

note that, then, $\pi_1(y)$ belong to some $(n-1)$ dimensional face of I. On the other hand, if $y \in E \cap \partial I$, we set

$$\pi_1(y) = y. \tag{3.63}$$

We then

$$\text{extend } \pi_1 \text{ on the whole of } I \text{ such that } \pi_1(I) \subset I \text{ and } \pi_1 \text{ is Lipschitz on } I. \tag{3.64}$$

(This can be done via standard extension results, see for example [H05, Theorem 2.5]). Note that this definition is coherent, in the sense that one can glue together the definition of π_1 on each $I \in \mathscr{C}_R^2(x, r)$ into a unique map π_1 defined on the whole of $D_R^2(x, r)$. Indeed, if $I, I' \in \mathscr{C}_R^2(x, r)$ are so that $I \cap I' \neq \emptyset$, then the definition of π_1 on $I' \cap I$ must agree, since $I \cap I'$ is contained in ∂I and $\partial I'$. Furthermore, we extend the definition of π_1 to $\mathbb{R}^n \setminus D_R^2(x, r)$ by setting

$$\pi_1(y) = y \tag{3.65}$$

there. Thus (3.62)-(3.65) give a coherent definition of π_1 on the whole of $\mathbb{R}^n$.

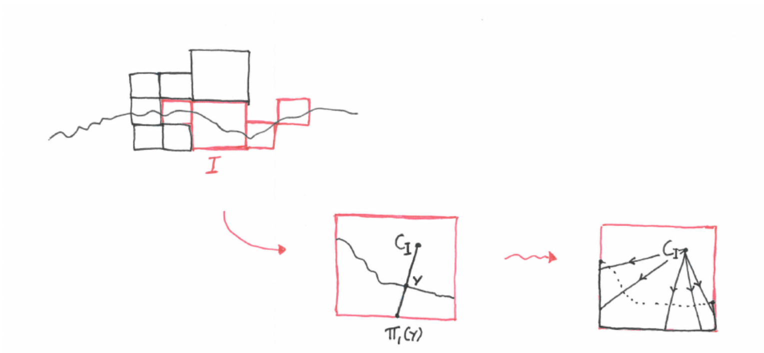

FIGURE 2. The first step in the construction of Federer-Fleming projections.

Now, if $d = n-1$, we stop here and we set $\varphi := \pi_1$. Otherwise, we continue as follows. We want to send points on the $(n-1)$-dimensional faces of cubes in $\mathscr{C}_R(x, r)$ to the boundaries of these faces, which are, in turn, $(n-2)$-dimensional faces. To do this, we proceed, as above, by defining the map we need on each individual face. Recall the definition of $\mathscr{F}^{n-1}$ in (3.41). Let us start by defining π_2 on each $\partial T \cup (\pi_1(E) \cap T)$, where $T \in \mathscr{F}^{n-1}$: we repeat the construction above. Namely,

$$\text{we find a point } c_T \in \mathrm{Int}(T) \setminus \pi_1(E) \text{ and then project radially } \pi_1(E) \cap T \text{ onto } \partial T; \tag{3.66}$$

once again, this definition leave unchanged those points which already belong to ∂T. Moreover, we can extend π_2 as a Lipschitz map from T to T (for each $T \in \mathscr{F}^{n-1}$), again by standard extension results. In this way, we obtain a coherently defined map on ∂I, for each $I \in \mathscr{C}_R(x,r)$. Next,

$$\begin{array}{c}\text{we extend } \pi_2 \text{ to the whole of } D_R(x,r) \text{ by requiring that}\\ \pi_2(I) \subset I \text{ for any } I \in \mathscr{C}_R(x,r).\end{array} \tag{3.67}$$

To do so, we want to extend π_2 from ∂I, to the whole of I, with the requirement that $\pi_2(I) \subset I$. Let c_I be the center of I. We set $\pi_2(c_I) := x^*$, where x^* is any point in $\pi_2(\partial I)$. Then for any point $y \in \partial I$, and a point $x = tc_I + (1-t)y$, $t \in [0,1]$, (so that x belongs to the line segment from c_I to y), we set

$$\pi_2(x) = t\pi_2(c_I) + (1-t)\pi_2(y).$$

Note that, because both $\pi_2(c_I)$ and $\pi_2(y)$ belong to ∂I, and I is convex, then $\pi_2(x) \in I$. Let us check that π_2 so defined is Lipschitz on I. Take any two points $x_1, x_2 \in I$ and write them as

$$x_1 = tc_I + (1-t)y_1,\ t \in [0,1] \text{ and } y_1 \in \partial I; \tag{3.68}$$

$$x_2 = sc_I + (1-s)y_2,\ s \in [0,1] \text{ and } y_2 \in \partial I. \tag{3.69}$$

Assume first that $t = s$. We can assume that $t = s < 1$, for otherwise $x_1 = x_2$. In this case, we have that

$$\begin{aligned}|\pi_2(x_1) - \pi_2(x_2)| = |(1-t)(\pi_2(y_1) - \pi_2(y_2))| &\le |1-t|C|y_1 - y_2|\\ &= C\,|(1-t)(y_1 - y_2)|\\ &= |x_1 - x_2|.\end{aligned}$$

Here the constant C is the Lipschitz constant of π_2 as function defined on ∂I.

Next, let us suppose that for x_1 and x_2 as in (3.68) and (3.69), we have that $y_1 = y_2$, hence they lie on the same line segment from c_I to ∂I. We first note that (assuming without loss of generality that $t > s$),

$$|x_1 - x_2| = |(t-s)(c_I - y_1)| \ge (t-s)\ell(I).$$

On the other hand, simply beacuse both $\pi_2(c_I)$ and $\pi_2(y_1)$ belong to ∂I, we have that

$$|\pi_2(x_1) - \pi_2(x_2)| = |(t-s)(\pi_2(c_I) - \pi_2(y_1))| \le \sqrt{n}\,(t-s)\ell(I).$$

Thus $|\pi_2(x_1) - \pi_2(x_2)| \le \sqrt{n}|x_1 - x_2|$. Finally, for any two points $x_1, x_2 \in I$ as in (3.68) and (3.69), put

$$x_2' := t\,c_I + (1-t)y_2.$$

Note that there exists a constant, depending only on n, so that

$$|x_2 - x_2'| \le C\,|x_1 - x_2|, \tag{3.70}$$

But then, by the triangle inequality, we also have that

$$|x_1 - x_2'| \le (C+1)|x_1 - x_2|.$$

This give us the following:

$$\begin{aligned}|\pi_2(x_1) - \pi_2(x_2)| &\le |\pi_2(x_1) - \pi_2(x_2')| + |\pi_2(x_2') - \pi_2(x_2)|\\ &\le C\,(|x_1 - x_2'| + |x_2' - x_2|)\\ &\le C'|x_1 - x_2|.\end{aligned}$$

This proves that the extension of π_2 to the whole of I is indeed Lipschitz, with a Lipschitz constant comparable to that of π_2 as defined on ∂I. Now we let π_2 on $D(x,r)$ to be piecewise defined on each I of $\mathscr{C}_R(x,r)$.

Let us see why this definition is coherent. If $T, T' \in \mathscr{F}^{n-1}$, $T \cap T' \ne \emptyset$ and let us assume without loss of generality that $\ell(T') < \ell(T)$, then either

$$T' \subset T, \tag{3.71}$$

or

$$T \cap T' \subset (\partial T) \cup (\partial T'). \tag{3.72}$$

If (3.72) holds, than we immediately see that the definition of π_2 is coherent, since we defined to be the identity on both ∂T and $\partial T'$. We divert a moment from the main construction to show that the former case does not happen.

LEMMA 3.28. *The case* (3.71) *does not occur.*

PROOF. Let $T \in \mathscr{F}^{n-1}(x,r)$, and assume first that T is an $(n-1)$-dimensional face (as opposed to a tile) of a cube $I \in \mathscr{C}_R(x,r)$. Suppose that there exists an element T' of $\mathscr{F}^{n-1}(x,r)$ such that $T' \subset T$. If T' is an $(n-1)$-dimensional face of a cube $I' \in \mathscr{C}_R(x,r)$, then, by construction of $\mathscr{F}^{n-1}(x,r)$, we must have that $I' \in \mathrm{Adj}^{n-1}(T)$. But then F cannot possibly belong to $\mathscr{F}^{n-1}$. On the other hand, if T' is a tile, then also in this case T cannot be in $\mathscr{F}^{n-1}$, since it should have been tessellated into tiles of the same size of T'.

Suppose now that T is a tile itself. But by construction, we cannot have two tiles of different sizes lying on the same $(n-1)$-dimensional face. Thus $T' \subset T$ has to really be $T' = T$, which contradicts the fact that $\ell(T') < \ell(T)$. $\square$

Thus the definition of π_2 is coherent. Let us now define π_2 on those $(n-1)$-dimensional faces T' of cubes in $\mathscr{C}^2_R(x,r)$ such that $\mathrm{Int}(T') \nsubseteq \mathrm{Int}(D_R(x,r))$ (recall the definition of $D_R(x,r)$, (3.54)). These are the faces which form the external boundary of the sheath $\mathscr{C}^2_R(x,r) \setminus \mathscr{C}_R(x,r)$. For these faces we leave everything unchanged, i.e. we let

$$\pi_2(y) = y \text{ for any } y \in T, \tag{3.73}$$

$$\text{where } T \text{ is a } (n-1)\text{-dimensional face } T \text{ with } T \nsubseteq D_R(x,r). \tag{3.74}$$

Finally, we extend π_2 to the whole of $D^2_R(x,r) \setminus D_R(x,r)$ by requiring that

$$\pi_2(I) \subset I \text{ for } I \in \mathscr{C}^2_R(x,r) \setminus \mathscr{C}_R(x,r)$$

$$\pi_2(y) = y \text{ whenever } y \in \partial D^2_R(x,r). \tag{3.75}$$

(This can be done in the same fashion as for (3.67)). We finally set

$$\pi_2(y) = y \text{ whenever } y \in \mathbb{R}^n \setminus D^2_R(x,r). \tag{3.76}$$

Hence (3.66)-(3.76) give us a Lipschitz map π_2 defined on the whole of $\mathbb{R}^n$. Now, if $d = n-2$, then we can set $\varphi = \pi_2 \circ \pi_1$, otherwise we continue projecting. To do so, we define a third map π_3. We follow the procedure above: first, if F is an $(n-2)$-dimensional element of $\mathscr{F}^{n-2}(x,r)$, then we set π_3 to be the radial projection from some point $c_F \in \mathrm{Int}(F) \setminus \pi_2 \circ \pi_1(E)$ defined on $\partial F \cup (\pi_1 \circ \pi_2(E) \cap F)$. In particular, $\pi_3(y) = y$ if $y \in \partial F$. Next, we extend π_3 to the whole of T, by requiring that $\pi_3(T) \subset T$; if there is an element F of $\mathscr{F}^{n-2}$ such that $(\pi_2 \circ \pi_1)(E) \cap F = \emptyset$, we set $\pi_3(y) = y$ on such an element. Note that this definition is coherent by construction of $\mathscr{F}^{n-2}(x,r)$, as in the definition of π_2. Next, we extend the definition of π_3 to the faces T of dimension $(n-1)$, requiring that for any such a face, we have $\pi_3(T) \subset T$ and $\pi_3(\mathrm{Int}(T)) \subset \mathrm{Int}(T)$; we also require that $\pi_3(y) = y$ on those faces T such that $T \cap D_R(x,r) = \emptyset$. Finally, we extend π_3 to the whole cubes I, requiring again that $\pi_3(I) \subset I$. At this point, note that for $y \in E$, we have

- either $\pi_3 \circ \pi_2 \circ \pi_1(y) \in \mathbb{R}^n \setminus D^2_R(x,r)$ if $y \in E \setminus D^2_R(x,r)$;
- or $\pi_3 \circ \pi_2 \circ \pi_1(y) \in T$, where T is a $(n-1)$-dimensional face of a cube in $\mathscr{C}^2_R(x,r)$ s.t. $T \nsubseteq D_R(x,r)$;
- or $\pi_3 \circ \pi_2 \circ \pi_1(y) \in F$, where $F \in \mathscr{F}^{n-3}$.

REMARK 3.29. The second possibility only occurs for those $y \in E$ so that

$$y \in \bigcup_{I \in \mathscr{C}^2(x,r) \setminus \mathscr{C}(x,r)} I \subset \mathbb{R}^n \setminus B(x,r).$$

We continue constructing projections in this fashion until reaching the d-dimensional skeleton. At each step, we construct π_m, for $n-d \leq m \leq n$, first on the elements of $\mathscr{F}^m(x,r)$ as a radial projection, and second we extend this definition to faces (or tiles) of increasing dimension, asking (if F' represents on such face or tile) that $\pi_m(F') \subset F'$. We stop once π_{n-d} has been defined. If $y \in E$, then, setting

$$\pi := \pi_{n-d} \circ \cdots \circ \pi_1, \tag{3.77}$$

we see that

- either $\pi(y) \in \mathbb{R}^n \setminus D_R^2(x,r)$, if $y \in E \setminus D_R^2(x,r)$;
- or $\pi(y) \in T$, where T is an $(n-1)$-dimensional face of a cube in $\mathscr{C}_R^2(x,r)$ such that $T \nsubseteq D_R(x,r)$;
- or $\pi(y) \in F$, where $F \in \mathscr{F}^d(x,r)$.

Note that the definition of π is coherent for the same reasons that π_3 and π_2 were coherent. In particular, π is Lipschitz (with possible a very large Lipschitz constant, but we do not mind this). Moreover, it follows from the construction that the properties (3.59), (3.60) and (3.61) are satisfied; this concludes the proof of Lemma 3.27

7. TC implies STC

In this section, we will prove that the topological condition (TC), imposed on E, implies the condition STC on the approximating sets E_R (see in Definition 3.25, see (3.43)-(3.50)). Our proof follows the proof of [**Dav04**], pages 200-202, with some changes to adapt it to the current situation.

LEMMA 3.30. *Let $E \subset \mathbb{R}^n$ be such that $0 < \mathcal{H}^d(E) < \infty$. Suppose moreover that E satisfies the $(r_0, \alpha_0, \delta_0, \eta_0)$-(TC), for some given parameteres $r_0, \alpha_0, \delta_0, \eta_0$ and let $Q_0 \in \mathcal{D}(E)$ be such that $\ell(Q_0) < r_0$. For some $k_0 \in \mathbb{N}$, apply Lemma 2.4 to Q_0 to obtain a corona decomposition $\mathsf{Top}(k_0) = \mathsf{Top}(Q_0, k_0)$ and a family of sets $\{E_R\}_{R \in \mathsf{Top}(k_0)}$ with parameter τ. Then we can find parameters r_1, α_1, δ_1 and η_1, so that the family $\{E_R\}_{R\in\mathsf{Top}(k_0)}$ satisfies the C_2-(STC) for C_2 sufficiently large.*

We will prove this lemma through a few lemmata below.

Set

$$\tau < \frac{1}{1000} \min\{\alpha_0, \eta_0\}. \tag{3.78}$$

Now, let $\mathsf{Top} = \mathsf{Top}(k_0)$; recall that for a large constant C_2, we want to prove the existence of parameters $r_1 > 0$ and $0 < \alpha_1, \delta_1 < 1$ (as in (3.43)) so that for all $x_1 \in E$, $R \in \mathsf{Top}$ and $Q \in \mathsf{Tree}(R)$ with $x_1 \in Q$, as in (3.44)-(3.45), for which (3.47) holds, we have the lower bound (3.49). Let us immediately choose the parameters in (3.43) (our choice is that of [**Dav04**], equation 3.10). We set

$$r_1 = r_0, \text{ where } r_0 \text{ is the one given by (TC)}; \tag{3.79}$$

$$\alpha_1 = C \min(\eta_0, \alpha_0); \tag{3.80}$$

$$\eta_1 = C\frac{\delta_0}{C_2}; \tag{3.81}$$

$$\delta_1 = C\delta_0. \tag{3.82}$$

We will fix the various absolute constants C as we go along. They will only depend on n. Let x_1, R and Q as in (3.44), (3.45) and (3.46). We now want to find a ball $B(x_2, r_2)$ with the required properties. We choose

$$x_2 = x_1 \text{ and} \tag{3.83}$$

$$r_2 \text{ s.t. } \frac{\ell(Q)}{3} \leq r_2 \leq \frac{2\ell(Q)}{3}. \tag{3.84}$$

REMARK 3.31. We would also like the quantity $\mathcal{H}^d\left(E_R \cap A_{\eta_1 \ell(Q)}(x_2, r_2)\right)$ to be small. Indeed, if for some choice of r_2 it held that

$$\mathcal{H}^d\left(E_R \cap A_{\eta_1\ell(Q)}(x_2, r_2)\right) \lesssim \delta_1 \ell(Q)^d, \tag{3.85}$$

then in order to verify (3.49) (adjusting the constant in the definition of δ_1), we would only have to check that

$$\mathcal{H}^d\left(\varphi_1(E_R \cap B(x_2, r_2))\right) \geq \delta_1 \ell(Q)^d. \tag{3.86}$$

CLAIM 1. *Such a choice of r_2 is indeed possible.*

PROOF. Let $s_1, ..., s_N$ be a family of radii such that each one of them satisfies (3.84) and $\left\{A_{\eta_1\ell(Q)}(x_2, s_k)\right\}_{k=1}^N$ is a pairwise disjoint family of (concentric) annuli. By the definition of

$A_{\eta_1\ell(Q)}(x_2,r_2)$, we have that[1]

$$N \gtrsim \frac{1}{\eta_1}. \tag{3.87}$$

Moreover, because these annuli are pairwise disjoint, we see that

$$\sum_{k=1}^{N} \mathcal{H}^d(E_R \cap A_{\eta_1\ell(Q)}(x_2,s_k)) \le \mathcal{H}^d\left(E_R \cap B(x_2,\ell(Q))\right) \overset{(3.47)}{\le} C_2\ell(Q)^d.$$

Then by the pigeonhole principle and (3.81) and (3.82), we must have that for some $1 \le k \le N$,

$$\mathcal{H}^d(E_R \cap A_{\eta_1\ell(Q)}(x_2,s_k)) \le N^{-1}C_2\ell(Q)^d \overset{(3.87)}{\lesssim} \eta_1 C_2\ell(Q)^d \overset{(3.81)}{\approx} \delta_0\ell(Q)^d \overset{(3.82)}{\approx} \delta_1\ell(Q)^d.$$

Thus (3.85) holds putting $r_2 = s_k$. □

Lemma 3.32. *Let $I \in \mathscr{C}_R(x_2,r_2)$, where $\mathscr{C}_R(x,r)$ is defined in (3.52); also, here $x_2 = x_1 \in Q$, for some $Q \in \mathsf{Tree}(R)$, and r_2 is as in (3.84). Then*

$$\ell(I) \le C\,\tau\,\ell(Q). \tag{3.88}$$

Proof. It suffices to prove the lemma for $Q \in \mathsf{Stop}(R)$. If $I \in \mathscr{C}_R$, then we know that $\ell(I) \lesssim \tau d_R(I)$ (recall that τ depends on α_0 and η_0 and was fixed in (3.78)). Moreover, $d_R(\cdot)$ is 1-Lipschitz. Then if $y \in I \cap B(x_2,r_2)$, we have that

$$d_R(I) \le d_R(y) \le |y - x_2| + d_R(x_2).$$

Because $r_2 \sim \ell(Q)$, $|x_2 - y| \le \ell(Q)$. On the other hand, we see that

$$d_R(x_2) = \inf_{P \in \mathsf{Stop}(R)} (\ell(P) + \operatorname{dist}(x_2,P)) \le \ell(Q).$$

□

Remark 3.33. Note that the same holds for any $I \in \mathscr{C}^2_R(x_2,r_2)$, by definition of $\mathscr{C}^2_R(x_2,r_2)$ (as defined in (3.55), (3.56) and (3.57)).

Lemma 3.34. *Let (x_2,r_2) to be as chosen in (3.83) and (3.84). For any one parameter family of Lipschitz deformations $\{\varphi_t\}_{0\le t\le 1}$ satisfying (3.1), (3.2), (3.3) and (3.48) (relative to (x_2,r_2)), the property (3.86) holds, that is, we have*

$$\mathcal{H}^d(\varphi_1(E_R \cap B(x_2,r_2))) \ge \delta_1\ell(Q)^d.$$

Proof. We have two ingredients we want to put together to achieve (3.86): on one hand, we know that something similar holds for E (i.e. TC); on the other hand, we know that E is locally well approximated by E_R, and we have a continuous (actually Lipschitz) way to move from E to E_R (i.e. the Federer-Fleming projection we constructed in the previous section). The idea is therefore the following: pick the one parameter family φ_t for which we want to show (3.86), and pick π as in Lemma 3.27. We will construct from these a deformation f which satisfies conditions (3.1)-(3.4); hence from (TC), we will deduce (3.86).

Set

$$\pi_t(y) := t\,\pi(y) + (1-t)\,y.$$

Note that if $y \in D^2_R(x_2,r_2)$, then (3.60) tells us that

$$|\pi_t(y) - y| \le t|\pi(y) - y| \le n^{1/2}\ell(I);$$

otherwise this quantity is equal to zero. By Lemma 3.32 we have that

$$\ell(I) \lesssim \tau d_R(y) \le \tau\left(|y - x_2| + d_R(x_2)\right) \lesssim \tau\ell(Q).$$

Hence we obtain

$$|\pi(y) - y| \lesssim \tau\ell(Q) \text{ for all } y \in \mathbb{R}^n. \tag{3.89}$$

[1]Indeed, if we let $s_k = \frac{\ell(Q)}{3}(1 - k\eta_1)$, we see that the annuli $A_{\eta_1\ell(Q)}(x_2,s_k)$ are pairwise disjoint. Moreover, we have $s_N = \frac{2}{3}$ whenever $N \approx \frac{1}{\eta_1}$.

Let us now define $\{f_t\}_{0\le t\le 1}$. We set

$$f_t(y) := \begin{cases} \pi_{2t}(y) & \text{if } 0 \le t \le \frac{1}{2}; \\ \varphi_{2t-1}(\pi(y)) & \text{if } \frac{1}{2} \le t \le 1. \end{cases} \tag{3.90}$$

We claim that $\{f_t\}$ satisfies the conditions (3.1)-(3.4) applied to the larger ball

$$B(x_2, \widetilde{r}) \text{ where } \widetilde{r} := (1+\eta_0)r_2. \tag{3.91}$$

We verify these conditions one by one. It is immediate from the definition that each f_t is Lipschitz.

Claim 2. *We have that $f_t(\bar{B}(x,\widetilde{r})) \subset \overline{B}(x,\widetilde{r})$, i.e. (3.1) holds for f_t.*

Proof of claim 2. Note that

$$B(x_2, r_2) \cap E \subset D_R(x_2, r_2) \subset D_R^2(x_2, r_2), \tag{3.92}$$

where $D_R(x_2,r_2)$ and $D_R^2(x_2,r_2)$ were defined in (3.54) and (3.58); (3.92) follows immediately from the definitions. Moreover, using Lemma 3.32, we see that any cube which was added to $\mathscr{C}_R^2(x_2,r_2) \setminus \mathscr{C}_R(x_2,r_2)$, must have side length at most $C\tau\ell(Q)$ (recall that Q satisfies (3.46) and x_2 and r_2 are as in (3.83) and (3.84), respectively). Thus $D_R^2(x_2,r_2) \subset B(x_2, r_2 + C\tau\ell(Q))$ and also, since

$$\tau\ell(Q) \overset{(3.78)}{\le} \frac{1}{100}\eta_0\ell(Q) \overset{(3.84)}{\le} \eta_0 r_2,$$

we have that

$$D_R^2(x_2,r_2) \subset B(x_2, r_2+\tau\ell(Q)) \subset B(x_2,(1+\eta_0)r_2) \subset B(x_2,\widetilde{r}), \tag{3.93}$$

Let us consider a few cases separately.

- If $y \in D_R^2(x_2,r_2)$, then $\pi_t(y) \in D_R^2(x_2,r_2)$ for all $0 \le t \le 1$ (by Lemma 3.27), and so
$$f_s(y) \in B(x_2,\widetilde{r}) \text{ for } s \le \frac{1}{2} \text{ whenever } y \in D_R^2(x_2,r_2).$$
Now, recall that (3.1) holds for φ_t (relative to $B(x_2,r_2)$); hence if, together with $y \in D_R^2(x_2,r_2)$, we also have that $\pi(y) \in \overline{B}(x_2,r_2)$, we then conclude that $f_s(y) \in \overline{B}(x_2,r_2)$ for $\frac{1}{2} \le s \le 1$. Also (3.3) holds for φ_t: if $\pi(y) \notin \overline{B}(x_2,r_2)$, then, for $\frac{1}{2} \le s \le 1$ and recalling (3.92), $f_s(y) = \pi(y) \in D_R^2(x_2,r_2)$. We obtain then that
$$f_s(y) \in B(x_2,\widetilde{r}) \text{ whenever } y \in D_R^2(x_2,r_2). \tag{3.94}$$
- Suppose now that $y \in B(x_2,\widetilde{r}) \setminus D_R^2(x_2,r_2)$; by construction $\pi(y) = y$ whenever $y \in \mathbb{R}^n \setminus D_R^2(x_2,r_2)$, hence $\pi_t(y) = y$ for $t \in [0,1]$, and thus
$$f_s(y) = y \text{ for } s \le \frac{1}{2} \text{ whenever } y \in B(x_2,\widetilde{r}) \setminus D_R^2(x_2,r_2). \tag{3.95}$$
But because φ_t satisfies (3.3) relative to $B(x_2,r_2)$ (that is, φ_t is the identity outside $B(x_2,r_2)$, and becuase $B(x_2,r_2) \subset D_R^2(x_2,r_2)$ (i.e. (3.92)), then $\varphi_t(\pi(y)) = \pi(y) = y$. Thus we obtain that
$$f_s(y) \in B(x_2,\widetilde{r}) \text{ for } \frac{1}{2} \le s \le 1 \text{ whenever } y \in B(x_2,\widetilde{r}) \setminus D_R^2(x_2,r_2). \tag{3.96}$$
Now (3.94), (3.95) and (3.96) give us the property (3.1) for $\{f_s\}$ relative to $B(x_2,\widetilde{r})$. □

Claim 3. *The path $s \mapsto f_s(y)$ is continuous, that is, (3.2) holds for f_t. Condition (3.3) also holds.*

Proof of claim 3. The first conclusion is clear, since $t \mapsto \pi_t(y)$ is continuous; moreover, $\pi_1(y) = \pi(y) = \varphi_0(\pi(y)) = f_{\frac{1}{2}}(y)$, and $t \mapsto \varphi_{2t-1}(\pi(y))$ is also continuous. As for the second conclusion of the claim, we see that $f_0(y) = \pi_0(y) = y$; if $y \in \mathbb{R}^n \setminus B(x_2,\widetilde{r})$, we have seen above that $f_t(y) = y$. Thus (3.3) holds for $\{f_s\}$ in $B(x_2,\widetilde{r})$. □

Claim 4. *Condition (3.4) holds, i.e. we have that* $\operatorname{dist}(f_s(y),E) \le \alpha_0\widetilde{r}$ *for $s \in [0,1]$ and $y \in E \cap B(x_2,\widetilde{r})$.*

Proof of claim 4. First, consider $0 \le s \le \frac{1}{2}$; let $y \in E \cap B(x_2, \widetilde{r})$; then we have

$$\begin{aligned} \operatorname{dist}(f_s(y), E) &= \operatorname{dist}(\pi_{2s}(y), E) \le |\pi_{2s}(y) - y| \\ &\overset{(3.89)}{\le} n^{\frac{1}{2}} \tau \ell(Q) \overset{(3.78)}{\le} \frac{1}{3}\alpha_0 \ell(Q) \overset{(3.84),(3.91)}{\le} \alpha_0 \widetilde{r}. \end{aligned}$$

Now suppose that $s > \frac{1}{2}$. If $y \in B(x_2, \widetilde{r}) \cap E$, then,

$$\text{either } y \in D_R^2(x_2, r_2) \tag{3.97}$$

$$\text{or } y \notin D_R^2(x_2, r_2). \tag{3.98}$$

If y is so that (3.98) holds, then $\pi(y) = y$, and moreover, from (3.3) for φ_t relative to $B(x_2, r_2)$, we see that $\varphi_{2s-1}(\pi(y)) = \varphi_{2s-1}(y) = y$. Hence (3.4) holds in this case. On the other hand, suppose that (3.97) holds. Then from the proof of Lemma 3.27, we have that

$$\text{either } \pi(y) \in E_R, \tag{3.99}$$

$$\text{or } \pi(y) \in T, \text{ where } T \text{ is an } (n-1)\text{-face with } T \nsubseteq D_R(x_2, r_2). \tag{3.100}$$

If (3.99) holds, then $\operatorname{dist}(\varphi_{2s-1}(\pi(y), E) \le \alpha_1 \ell(Q)$ by (3.48) applied to φ_t; this immediately implies $\operatorname{dist}(f_s(y), E) \le \alpha_0 \widetilde{r}$ for $s > \frac{1}{2}$ by the choice of α_1 in (3.80). On the other hand, if (3.100) holds, we must have $\pi(y) \notin B(x_2, r_2)$ (by construction of π), and therefore $\varphi_{2s-1}(\pi(y)) = \pi(y)$. Now $\pi(y)$ belongs to a cube in $\mathscr{C}_R^2(x_2, r_2)$ with side length at most $\tau\ell(Q)$ and touching E, hence we retrieve $\operatorname{dist}(f_s(y), E) \le \alpha_0 \ell(Q) \le \alpha_0 \widetilde{r}$. Together with the previous estimates, we obtain that $\{f_s\}$ satisfies (3.4) for all $s \in [0, 1]$. This concludes the proof of claim 4. □

Claim 2-4 now show us that f is an allowed Lipschtiz deformation relative to the ball $B(x_2, \widetilde{r})$, with $\widetilde{r} = (1 + \eta_0) r_2$. Thus, by the topological condition (TC), we have

$$\mathcal{H}^d \left(B(x_2, (1 - \eta_0)\widetilde{r}) \cap f_1(E) \right) \ge \delta_0 \widetilde{r}^d. \tag{3.101}$$

Since $B(x_2, r_2) \supset B(x_2, (1 - \eta_0)\widetilde{r})$, (3.101) implies that

$$\mathcal{H}^d \left(B(x_2, r_2) \cap f_1(E) \right) \ge \delta_0 (1 + \eta_0)^d r_2^d. \tag{3.102}$$

Recall from above that if $y \in E$ and it is such that $y \notin D_R(x_2, r_2)$, then $f_1(y) \notin B(x_2, r_2)$. Thus,

$$B(x_2, r_2) \cap f_1(E) \subset f_1(D_R(x_2, r_2) \cap E).$$

Note also that $\pi(E \cap D_R(x_2, r_2)) \subset E_R$. Thus we obtain, using (3.102), (3.85) and (3.82),

$$\mathcal{H}^d \left(\varphi_1(E_R) \cap B(x_2, r_2) \right) \ge 2\delta_1 \ell(Q)^d.$$

This concludes the proof of Lemma 3.34. □

Proof of Lemma 3.30. Recall from Remark 3.31, that proving the estimates (3.85) and (3.86) is enough to prove (3.49), and thus Lemma 3.30. Claim 1 gives the first estimate, while Lemma 3.34 gives the second one. □

8. A further approximating set

We now construct a dyadic approximation of E_R. We will then show that this approximation satisfies the STC; in the next section, we will show that this implies that this dyadic approximation has large intersection with a uniformly rectifiable set.

Let ρ be a small parameter (which we will fix later, and can be assumed to be of the form $2^{-k}, k \in \mathbb{N}$). We write

$$\Delta_\rho := \Delta_{j(\rho)},$$

where $j(\rho)$ is an integer so that $2^{-j(\rho)} = \rho$.

We set

$$\mathscr{C}_{R,\rho} := \{ I \in \Delta_\rho \,|\, I \cap E_R \neq \emptyset \}; \tag{3.103}$$

$$E_\rho = E_{R,\rho} := \bigcup_{I \in \mathscr{C}_{R,\rho}} \partial_d I. \tag{3.104}$$

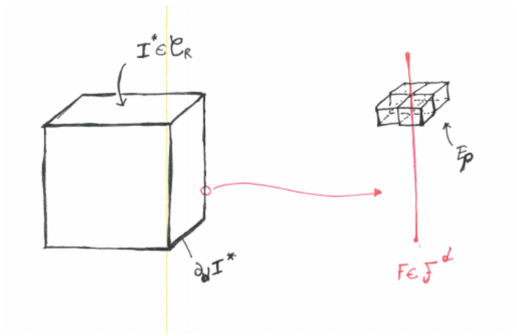

FIGURE 3. How the set $E_{R,\rho}$ is constructed.

LEMMA 3.35. *Let I_* be the smallest cube in $\mathscr{C}_R$ (which exists since $\mathscr{C}_R$ is finite). Then for all $\rho < \ell(I_*)$, $E_{R,\rho}$ is Ahlfors regular, with regularity constant comparable to that of E_R.*

PROOF. Let T be a d-dimensional face of some cube $J \in \mathscr{C}_R$. Denote by $\mathscr{F}^d_\rho$ the collection of d-dimensional faces from cubes in $\mathscr{C}_{R,\rho}$. Then we can cover T with a subcollection of $\mathscr{F}^d_\rho$ with pairwise disjoint interiors. If we denote such a collection by $\mathscr{F}^d_\rho(T)$, then it is obvious that

$$\mathcal{H}^d(T) = \sum_{F \in \mathscr{F}^d_\rho} \mathcal{H}^d(F).$$

To each such a face $F \in \mathscr{F}^d_\rho$, there corresponds a bounded number of cubes so that $F \subset I \in \mathscr{C}_{R,\rho}$. This bounded number depends only on n and d. Moreover, each of these cubes has a bounded number of other d-dimensional faces, and, again, this number depends only on n and d. Thus, if we denote by $\Delta_\rho(T)$ the family of cubes in Δ_ρ which also meet T, we see that

$$\sum_{I \in \Delta_\rho(T)} \mathcal{H}^d(I) \leq C(n,d) \sum_{F \in \mathscr{F}^d_\rho} \mathcal{H}^d(F) = C(n,d)\, \mathcal{H}^d(T).$$

Then, we see that

$$\begin{aligned}
\mathcal{H}^d\left(E_{R,\rho} \cap B(x,r)\right) &\leq \sum_{\substack{I \in \mathscr{C}_{R,\rho} \\ I \cap B(x,r) \neq \emptyset}} \mathcal{H}^d(\partial_d I) \\
&\leq \sum_{J \in \mathscr{C}_R(x,r)} \sum_{T \text{ face of } J} \sum_{I \in \mathscr{F}^d_\rho(T)} \mathcal{H}^d(\partial_d I) \\
&\leq C(n,d) \sum_{J \in \mathscr{C}_R(x,r)} \mathcal{H}^d(\partial_d J) \\
&\lesssim \mathsf{C}_3 r^d.
\end{aligned}$$

Recall that $\mathscr{C}_3$ is the upper regularity constant of E_R (see Notation 3.24). By enlarging it if necessary, we will assume that C_3 is also the regularity constant of $E_{R,\rho}$. Lower regularity is even easier since we are adding mass. □

LEMMA 3.36. *Lemma 3.30 holds when we substitute E_R with $E_{R,\rho}$.*

PROOF. Let C_2 be the same constant as in Lemma 3.30. We know from Section 7, that E_R satisfies the STC with a choice of constants as in (3.79) - (3.82). Let (x_2, r_2) as in (3.83) and (3.84). We now add to the constraint on ρ given in the statement of Lemma 3.35, the following one: we ask that

$$\rho < \frac{1}{1000\sqrt{n}} \min\{\eta_1, \alpha_1\}\, \ell(Q) \text{ for all } Q \in \mathsf{Tree}(R). \tag{3.105}$$

Note that because $E_{R,\rho}$ is Ahlfors regular independently of ρ, this does not cause any trouble. Moreover, we can always choose $\rho > 0$ since $\mathsf{Tree}(Q)$ is a finite family.

Let $Q \in \mathsf{Tree}(R)$ and (x_2, r_2) as in the proof of Lemma 3.30, in particular see (3.83) and (3.84). Note that $E_{R,\rho} \cap B(x_2, r_2) \supset E_R \cap B(x_2, r_2)$, simply because $E_R \subset E_{R,\rho}$. Thus, also

$\varphi_1(E_{R,\rho} \cap B(x_2, r_2)) \supset \varphi_1(E_R \cap B(x_2, r_2))$, and therefore

$$\begin{aligned}\mathcal{H}^d(\varphi_1(E_{R,\rho} \cap B(x_2, r_2))) &\geq \mathcal{H}^d\left(\varphi_1(E_R \cap B(x_2, r_2))\right) \\ &\geq \delta_1 \ell(Q)^d + \mathcal{H}^d(E_R \cap A_{\eta_1 \ell(Q)}(x_2, r_2)).\end{aligned}$$

But now note that if we choose a parameter $c > 0$ sufficiently small, and we put $\widetilde{\eta}_1 = c\eta_1$, then we see that

$$\mathcal{H}^d(E_R \cap A_{\eta_1 \ell(Q)}(x_2, r_2)) \geq \mathcal{H}^d(E_{R,\rho} \cap A_{\widetilde{\eta}_1 \ell(Q)}(x_2, r_2)).$$

Note that c only depends on n and d. Hence we obtain that

$$\mathcal{H}^d(\varphi_1(E_{R,\rho} \cap B(x_2, r_2))) \geq \delta_1 \ell(Q)^d + \mathcal{H}^d(E_{R,\rho} \cap A_{\widetilde{\eta}_1 \ell(Q)}(x_2, r_2)), \tag{3.106}$$

and the lemma is proven. □

Remark 3.37. To ease the notation, we ignore the fact that we changed η_1 by a constant c, and we will continue denoting $\widetilde{\eta}_1$ by η_1.

9. STC implies that $E_{R,\rho}$ has large intersections with some uniformly rectifiable set

In this section, we show that the topological condition STC imposed on E_R (and thus on $E_{R,\rho}$) tells us that $E_{R,\rho}$ has a large intersection (with repsect to the scale of each cube $Q \in \mathsf{Tree}(R)$) with a uniformly d-rectifiable set. The idea is to define a functional whose minimizer F has large intersection with $E_{R,\rho}$. In turn F, by virtue of being a minimiser of such a functional, will turn out to be a quasiminimiser (in the sense of [**DS00**]), and thus uniformly rectifiable.

Remark 3.38. Once again, we follow David in [**Dav04**] and we adapt his proof to our current situation. The adaptations needed are fairly small, but annoying enough to justify the inclusion of the proofs.

9.1. Definition of a functional J. Let C_2 be a large constant (depending on C_3, the Ahlfors regularity constant of $E_{R,\rho}$ and E_R) and k_0 a sufficiently large integer. Then STC gives us constants $r_1 < r_0$, and $0 < \alpha_1, \eta_1, \delta_1 < 1$ (as in (3.43)) such that for every choice of $x_1 \in E$, $R \in \mathsf{Top}(k_0)$ and $Q \in \mathsf{Tree}(R)$ as in (3.44), (3.45) and (3.46) respectively, with also $\mathcal{H}^d(E_R \cap B(x_1, \ell(Q))) \leq C_2 \ell(Q)^d$, we can find a ball $B(x_2, r_2) \subset B(x_1, \ell(Q))$ for which, given an appropriate one-parameter family of Lipschitz deformations $\{\varphi_t\}$, we have the lower bound (3.49). From the previous sections, we see that this holds for both E_R and $E_{R,\rho}$.

Let us start by define the functional mentioned above.

Fix a cube $Q \in \mathsf{Tree}$ and let (x_2, r_2) be as in (3.83) and (3.84). To simplify the notation, we put

$$r := r_2; \tag{3.107}$$

$$x := x_2. \tag{3.108}$$

For later use, let us set

$$B_j := B\left(x, r - \eta_1 \ell(Q) + \frac{j \eta_1 \ell(Q)}{10}\right) \text{ for } 0 \leq j \leq 10. \tag{3.109}$$

Note that $B_{10} = B(x,r)$ and $B_{10} \setminus B_0 = A_{\eta_1 \ell(Q)}(x,r)$.

Recall the constraint on ρ, (3.105), and consider a constant σ (let it be a power of 2), so that

$$\frac{1}{200\sqrt{n}} \min\{\eta_1, \alpha_1\} \ell(Q) \leq \sigma \leq \frac{1}{100\sqrt{n}} \min\{\eta_1, \alpha_1\} \ell(Q). \tag{3.110}$$

Then we put

$$\Delta_\sigma(B_6) := \left\{ I \in \Delta_{j(\sigma)} \,|\, I \cap B_6 \neq \emptyset \right\}; \tag{3.111}$$

$$\mathscr{C}^1_\sigma(B_6) := \{ I \in \Delta_\sigma(B_6) \,|\, I \cap J \neq \emptyset \text{ for some } J \in \mathscr{C}_R \}; \tag{3.112}$$

$$\mathscr{C}^2_\sigma(B_6) := \left\{ I \in \Delta_\sigma(B_6) \,|\, \text{ there exists a } J \in \mathscr{C}^1_\sigma(B_6) \text{ with } I \cap J \neq \emptyset \right\}. \tag{3.113}$$

Finally we put

$$V_Q^1 := \bigcup_{I \in \mathscr{C}_\sigma^1(B_6)} I \quad \text{and} \tag{3.114}$$

$$V_Q^2 := \bigcup_{I \in \mathscr{C}_\sigma^2(B_6)} I. \tag{3.115}$$

Note that for any cube $I \in \mathscr{C}_R(B_6)$, there exists a cube $J \in \mathscr{C}_\sigma(B_6)$ so that $J \supset I$.

LEMMA 3.39. *With the notation above, we have that*

$$B_6 \cap E \subset V_Q^2 \subset B_7 \text{ and} \tag{3.116}$$

$$\operatorname{dist}(y, E) \leq \alpha_1 \ell(Q) \text{ for all } y \in V_Q^1. \tag{3.117}$$

PROOF. The first inclusion in (3.116) is immediate[2]. To see the second one, note that for any point $y \in V_Q^2$, we have that $\operatorname{dist}(y, V_Q^1) \leq \sqrt{n}\sigma$. In turn, any point in $V_Q^1 \setminus B_6$ can be at most $\sqrt{n}\sigma$ away from ∂B_6. Hence, by the choice of σ in (3.110), we see that if $y \in V_Q^2 \setminus B_6$, $\operatorname{dist}(y, \partial B_6) < \frac{1}{10}\eta_1 \ell(Q)$, and so (3.116) will be satisfied.

To prove (3.117), suppose first that $y \in V_Q^1$ is such that $y \in I$ for some $I \in \mathscr{C}_R$. Now, by construction, if $y \in V_Q^1$, then $y \in B(x_2, r_2)$, with $x_2 \in Q$ and $r_2 \sim \ell(Q)$. Thus $\operatorname{dist}(I, Q) \lesssim \ell(Q)$. This implies, using Lemma 3.20 (1), that

$$\ell(I) \leq \tau d_R(I) \leq \tau(\ell(Q) + \operatorname{dist}(I, Q)) \lesssim \tau \ell(Q).$$

But recall also that $\tau << \alpha_0, \eta_0$ (as in (3.78)), and thus also $\tau << \alpha_1$, by the choice of α_1 in (3.80). Since $I \cap E \neq \emptyset$, for otherwise I would not be in $\mathscr{C}_R$, we obtain that $\operatorname{dist}(y, E) \leq \alpha_1 \ell(Q)$ for these y. Note that if $y \in V_Q^1$ but $y \notin I$ for all $I \in \mathscr{C}_R$, then by the way that V_Q^1 was defined, $\operatorname{dist}(y, I) < \sigma$ for some $I \in \mathscr{C}_R$. Keeping in mind the definition of σ in (3.110), we can repeat the argument above to obtain (3.117) also in this case. □

We are now ready to fix the class of subsets upon which the said functional will be allowed to act. We set

$$\mathcal{F}_0 \tag{3.118}$$

to be the class of subsets F of $\mathbb{R}^n$ such that

$$F \text{ is closed (in the topology of } \mathbb{R}^n). \tag{3.119}$$

$$F \subset V_Q^2 \tag{3.120}$$

$$F = F^* \cup L. \tag{3.121}$$

Here L denotes any subset of Hausdorff dimension smaller or equal than $d-1$; by F^* we mean a finite union of d-dimensional faces of cubes coming from Δ_ρ. We will call F^* the *coral* part of F. In other words the class $\mathcal{F}_0$ is composed by subsets that are built out of a finite number of d-faces coming from cubes in Δ_ρ.

Let us consider a subclass of $\mathcal{F}_0$: we set

$$\mathcal{F} := \left\{F \in \mathcal{F}_0 \,|\, F = \varphi_1(E_{R,\rho} \cap V_Q^2)\right\}, \tag{3.122}$$

where $\{\varphi_t\}_{0 \leq t \leq 1}$ is a family of Lipschitz mappings on $\mathbb{R}^n$ such that

$$\varphi_t(V_Q^2) \subset V_Q^2 \text{ for all } t \in [0,1]; \tag{3.123}$$

$$t \mapsto \varphi_t(y) \text{ is continuous for all } y \in \mathbb{R}^n; \tag{3.124}$$

$$\varphi_t(y) = y \text{ for } t = 0 \text{ and for } y \in \mathbb{R}^n \setminus V_Q^2; \tag{3.125}$$

$$\operatorname{dist}(\varphi_t(y), E_R) < \alpha_1 \ell(Q) \text{ for } y \in E_{R,\rho} \cap V_Q^2 \text{ and all } t \in [0,1]; \tag{3.126}$$

$$\varphi_1(y) \in V_Q^1 \text{ for } y \in E_{R,\rho} \cap V_Q^2. \tag{3.127}$$

LEMMA 3.40. *We have that $E_{R,\rho} \cap V_Q^2 \in \mathcal{F}$. In particular, the class $\mathcal{F}$ is nonempty.*

[2] Indeed, if $y \in E \cap B_6$, then, first there exists a cube $J \in \mathscr{C}_R$ which contains y. But then $J \cap B_6 \neq \varnothing$. Then, because $\Delta_\sigma(B_6)$ covers B_6, we find a cube $I \in \Delta_\sigma(B_6)$ which intersects J and $y \in J \cap I$. Clearly $J \in \mathscr{C}_\sigma^1(B_6)$, and thus $y \in V_Q^2$.

PROOF. We just take the trivial deformation $\varphi_t(y) = y$ for all y and t, so that (3.123), (3.124) and (3.125) hold immediately. Moreover, by construction we have that all points in $E_{R,\rho}$ are contained in a cube from $\mathscr{C}_{R,\rho}$. The side length of these cubes is (much) less than[3] $\alpha_1\ell(Q)$ and they must touch E_R. Hence $\text{dist}(y, E_R) < \alpha_1\ell(Q)$ and so (3.126) is satisfied. As far as condition (3.127) is concerned, we see that if $y \in E_{R,\rho} \cap V_Q^2$, then by definition of $E_{R,\rho}$ and $\mathscr{C}_{R,\rho}$ in (3.104) and (3.103), we see that y must lies in a cube which belongs to $\mathscr{C}_\sigma^1(B_6)$ (from the definition of σ in (3.110)), and thus it must be in V_Q^1. □

We now define the aforementioned functional. For some $c_2 < 1$ to be chosen later, we put

$$M = \frac{\mathsf{C}_3}{c_2\,\delta_1}, \tag{3.128}$$

where recall that C_3 is the Ahlfors regularity constant of E_R (as fixed in Notation 3.24) and of $E_{R,\rho}$. Then we set

$$J(F) := \mathcal{H}^d(F \cap E_{R,\rho}) + M\,\mathcal{H}^d(F \setminus E_{R,\rho}) \text{ for } F \in \mathcal{F}. \tag{3.129}$$

Note that $J(F) = J(F^*)$ (with notation as in (3.121)), and there is only a finite number of sets like F^*. Thus there exists a set $\widetilde{F} \in \mathcal{F}$ such that

$$J(\widetilde{F}) = \min_{F\in\mathcal{F}} J(F).$$

Note that, for a set $F \in \mathcal{F}$ trying to keep $J(F)$ small, it will be very expensive to have a large portion which does not intersect $E_{R,\rho}$, as M can be quite large. This is the reason why we expect the minimiser $\widetilde{F}$ to have a large intersection with $E_{R,\rho}$. This also implies that a minimiser of J also will lie close to E_R.

LEMMA 3.41. *Let $\widetilde{F}$ be a minimiser of J (as in (3.129)) in $\mathcal{F}$. Then*

$$\mathcal{H}^d(E_{R,\rho} \cap \widetilde{F}) \geq C\,\delta_1\ell(Q)^d. \tag{3.130}$$

Once again, the proof below comes from [**Dav04**], Section 4. To account for the small changes needed to adapt it to our situation, we include it nevertheless.

PROOF. Because $\widetilde{F} \in \mathcal{F}$, then $\widetilde{F} = \varphi_1(E_{R,\rho} \cap V_Q^2)$, where $\{\varphi_t\}$ satisfies (3.123)-(3.127). We want to check that this specific one parameter family φ_t satisfies also the conditions for the deformations used for STC (see Definition 3.25) relative to $B(x,r) = B(x_2, r_2)$. Note that $\{\varphi_t\}$ satisfies (3.1), (3.2) and (3.3), since from (3.116), we have that $V_Q^2 \subset B(x,r)$. We want to check that (3.48) holdd, that is

$$\text{dist}(\varphi_t(y), E_R) \leq \frac{\alpha_1\ell(Q)}{2} \text{ for } t \in [0,1] \text{ and } y \in E_{R,\rho} \cap B(x,r). \tag{3.131}$$

So, let $y \in E_{R,\rho} \cap B(x,r)$. If $y \notin V_Q^2$, then $\varphi_t(y) = y$ by (3.125); since $y \in E_{R,\rho}$, then by construction $\text{dist}(y, E_R) < \sqrt{n}\rho \ll \alpha_1\ell(Q)$ (by the constraint on ρ in (3.105)). Moreover, if $x \in E_R$ is the point closest to y, then $x \in \partial_d I$ for some $I \in \mathscr{C}_R(x,r)$ and, by Lemma 3.32, we have that $\ell(I) \lesssim \tau\ell(Q)$. All in all, this implies that $\text{dist}(y, E) < \alpha_1\ell(Q)$ whenever $y \in B(x,r) \cap E_{R,\rho}$ does not belong to V_Q^2.

If $y \in V_Q^2 \cap E_{R,\rho}$, then by (3.127), $\varphi_t(y)$ must lie in V_Q^1, and hence be at most σ far away from E_R (by construction); but $\sigma < \alpha_1\ell(Q)$, and repeating the argument of the previous case we again see that (3.131) holds.

Thus $\{\varphi_t\}$ is a one parameter family of Lipschitz deformations which satisfies the requirements stated in the definition of STC (see Definition 3.25). Recall that $E_{R,\rho}$ satisfies the STC (at scale $\ell(Q)$); we therefore have the lower bound (3.106), i.e.

$$\mathcal{H}^d(\varphi_1(E_{R,\rho} \cap B(x,r))) \geq \delta_1\ell(Q)^d + \mathcal{H}^d\left(E_{R,\rho} \cap A_{\widetilde{\eta}_1\ell(Q)}(x,r)\right).$$

Now, the family $\{\varphi_t\}$ which we are considering, not only satisfies (3.3), but also (3.125), and so, in particular,

$$\begin{aligned}\varphi_1\left(E_{R,\rho} \cap B(x,r)\right) &= \varphi_1\left((V_Q^2 \cap E_{R,\rho}) \cup (E_{R,\rho} \cap (B(x,r) \setminus V_Q^2))\right)\\ &= \varphi_1(E_{R,\rho} \cap V_Q^2) \cup \left(E_{R,\rho} \cap (B(x,r) \setminus V_Q^2)\right),\end{aligned}$$

[3]Since one such a cube will have side length ρ and $\rho \ll \min\{\eta_1, \alpha_1\}\ell(Q)$ from (3.110).

recalling that by definition (see (3.116)) $V_Q^2 \subset B(x,r)$. Also, note that

$$E_{R,\rho} \cap (B(x,r) \setminus V_Q^2) \subset A_{\widetilde{\eta}_1 \ell(Q)}(x,r) \cap E_{R,\rho}; \tag{3.132}$$

indeed, using $B_6 \cap E_{R,\rho} \subset V_Q^2$, we see that $E_{R,\rho} \cap (B(x,r) \setminus V_Q^2) \subset E_{R,\rho} \cap (B(x,r) \setminus B_6)$, and (recalling the definition of the balls B_j's (3.109)), $B(x,r) \setminus B_6 \subset A_{\widetilde{\eta}_1 \ell(Q)}(x,r)$. Thus we have

$$\mathcal{H}^d \left(\varphi_1(E_{R,\rho} \cap V_Q^2)\right) + \mathcal{H}^d \left(A_{\widetilde{\eta}_1 \ell(Q)}(x,r) \cap E_{R,\rho}\right) \geq \delta_1 \ell(Q)^d + \mathcal{H}^d \left(E_{R,\rho} \cap A_{\widetilde{\eta}_1 \ell(Q)}(x,r)\right)$$

and so

$$\mathcal{H}^d \left(\varphi_1(E_{R,\rho} \cap V_Q^2)\right) \geq \delta_1 \ell(Q)^d. \tag{3.133}$$

In particular, from the definition of $\mathcal{F}$, this inequality holds for any $F \in \mathcal{F}$. Recall now that we decided that $\widetilde{F}$ was a minimiser of J (as defined in (3.129)). Thus we have that

$$\begin{aligned} J(\widetilde{F}) &\overset{\text{Lemma 3.40}}{\leq} J(E_{R,\rho} \cap V_Q^2) = \mathcal{H}^d \left(E_{R,\rho} \cap V_Q^2\right) \\ &\overset{(3.116)}{\leq} \mathcal{H}^d \left(E_{R,\rho} \cap B(x,r)\right) \overset{\text{Lemma 3.35}}{\leq} \mathsf{C}_3 \ell(Q)^d. \end{aligned} \tag{3.134}$$

Moreover, by definition of J,

$$\mathcal{H}^d \left(\widetilde{F} \setminus E_{R,\rho}\right) \leq \frac{J(\widetilde{F})}{M} \overset{(3.134)}{\leq} \frac{C_3}{M} \ell(Q)^d \overset{(3.128)}{=} c_2\, \delta_1 \ell(Q)^d. \tag{3.135}$$

But then we have that, with c_2 sifficiently small,

$$\mathcal{H}^d \left(\widetilde{F} \cap E_{R,\rho}\right) = \mathcal{H}^d(\widetilde{F}) - \mathcal{H}^d(\widetilde{F} \setminus E_{R,\rho}) \overset{(3.133),(3.135)}{\geq} \frac{\delta_1 \ell(Q)^d}{2}. \tag{3.136}$$

This proves the Lemma. □

9.2. Almgren quasiminimality of $\widetilde{F}$. Roughly speaking, a set S in $\mathbb{R}^n$ is a quasiminimiser of the d-dimensional Hausdorff measure $\mathcal{H}^d$ if, whenever we deform S in a suitable way, the d-measure of such deformations does not shrink too much. Quasiminimality is a form of stability: the set maintain its Hausdorff dimension under a suitable class of perturbations. Heuristically, this is the reason why we need to transfer the topological condition from E to an Ahlfors regular set: in this case, modulo technicalities, quasiminimality roughly coincides with our topological condition.

We now recall from [**DS00**] the precise definitions to make this notion precise. Let U be an open set in $\mathbb{R}^n$ and fix two constants

$$1 \leq k < \infty \text{ and } 0 < \delta \leq +\infty. \tag{3.137}$$

Let $S \subset U$ be so that

$$S \neq \emptyset \text{ and } \overline{S} \setminus S \subset \mathbb{R}^n \setminus U. \tag{3.138}$$

Assume also that

$$\mathcal{H}^d(S \cap B) < +\infty \text{ for all balls } B \subset U. \tag{3.139}$$

Now, let us make precise what we mean by 'deformations' or 'perturbations'. Given a set S, deformations of S will be sets of the form $\phi(S)$, where

$$\phi : \mathbb{R}^n \to \mathbb{R}^n \text{ is Lipschitz} \tag{3.140}$$

and satisfies the following properties.

$$\operatorname{diam}(W \cup \phi(W)) \leq \delta \text{ where } W := \{x \in \mathbb{R}^n \mid \phi(x) \neq x\}; \tag{3.141}$$

$$\operatorname{dist}(W \cup \phi(W), \mathbb{R}^n \setminus U) > 0; \tag{3.142}$$

$$\phi \text{ is Lipschitz-homotopic to the identity.} \tag{3.143}$$

The last requirement means that there exists a continuous map

$$h : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$$

such that $h(x,0) = x$ and $h(x,1) = \phi(x)$ for all $x \in \mathbb{R}^n$, such that $h(\cdot,t) : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz for all $t \in [0,1]$, and such that

$$\operatorname{diam}(\widehat{W}) < \delta \text{ and } \operatorname{dist}(\widehat{W}, \mathbb{R}^n \setminus U) > 0,$$

where

$$\widehat{W} := \bigcup_{t\in[0,1]} W_t \cup \phi_t(W_t), \quad \phi_t(x) = h(x,t) \text{ and } W_t := \{x \in \mathbb{R}^n \mid \phi_t(x) \neq x\}.$$

Definition 3.42. Let $0 < d < n$; let $U \subset \mathbb{R}^n$ be an open set and fix two constant k, δ as in (3.137). We say that $S \subset U$ is a (U, k, δ)-*quasiminimizer* for $\mathcal{H}^d$ if S satisfies (3.138), (3.139) and

$$\mathcal{H}^d(S \cap W) \leq k\mathcal{H}^d(\phi(S \cap W)) \tag{3.144}$$

for all Lipschitz mappings ϕ which satisfy (3.141), (3.142) and (3.143).

Lemma 3.43. *The set*

$$S := \widetilde{F} \cap B_2 \tag{3.145}$$

is a (B_2, k, δ)*-quasiminimizer for* $\mathcal{H}^d$*, where*

$$k = C\, 4^{nd} M, \tag{3.146}$$

(here C *is a geometric constant), and*

$$\delta = c_3 \min\{\alpha_1, \eta_1\}\, \ell(Q). \tag{3.147}$$

Here $0 < c_3 < 1$ *is a parameter bounded above by a universal constant.*

We will need the following lemma from [**Dav04**]. We tailor it to our current notation.

Lemma 3.44 ([**Dav04**], Lemma 5.8). *Let* V^1 *be a finite union of dyadic cubes belonging to* Δ_σ*, where* σ *should be thought of as in* (3.110)*. There exists a* 4^n*-Lipschitz function* h*, defined on*

$$V_+^1 := \left\{ y \in \mathbb{R}^n \mid \operatorname{dist}(y, V^1) \leq \frac{\sigma}{4} \right\} \tag{3.148}$$

and such that

$$h(V_+^1) \subset V^1 \tag{3.149}$$

$$h(y) = y \text{ for } y \in V^1, \tag{3.150}$$

and

$$|h(y) - y| \leq n^{1/2}\sigma. \tag{3.151}$$

Recall that we want to show that $\widetilde{F}$ is a quasiminimal set for $\mathcal{H}^d$. Here is the idea to do so. We want to look at $\mathcal{H}^d(\phi(\widetilde{F} \cap W))$; what we know about $\widetilde{F}$ which makes us hope that it may well be a quasiminimal set is that $\widetilde{F}$ is a minimiser of the functional J as defined in (3.129). We want to use this information. In other words, we would like to say that $\phi(\widetilde{F})$ is a competitor of $\widetilde{F}$ belonging to the class $\mathcal{F}$. Unfortunately, this is not true, in the sense that $\phi(\widetilde{F})$ may lie outside V_Q^1, and this is not permitted (see (3.127)). What we can do however, is first to retract $\phi(\widetilde{F})$ (which we will call F_1) back into V_Q^1 (using the map h from Lemma 3.44); let us set $F_2 := h(F_1)$. Next, we want to project F_2 onto some d-dimensional skeleton so that it belongs to $\mathcal{F}_0$ (as defined in (3.118)). This projection will happen in two steps, with two corresponding maps; we will denote the images so obtained by F_3 and then F_4; this latter one will be the needed competitor. The last step will be to show that these distortions of $\phi(\widetilde{F})$ don't increase the size of $\phi(\widetilde{F})$ too much. In this way, first by the minimising property of $\widetilde{F}$ we will obtain a bound like $\mathcal{H}^d(\widetilde{F}) \leq M\mathcal{H}^d(F_4)$ and then, by this last step, a bound similar to $\mathcal{H}^d(F_4) \leq C\mathcal{H}^d(\phi(\widetilde{F}))$ and thus establishing quasiminimality.

9.2.1. *Consrtuction of* F_1 *and* F_2. Let us get started: we want to deform $\widetilde{F}$ with Lipshitz maps ϕ as in (3.140). Pick one such Lipshitz deformation ϕ. We are interested in those points $y \in W \cap \widetilde{F}$, i.e. those points which are actually being moved by ϕ. But by (3.141), we must have that $|\phi(y) - y| \leq \delta$. We put

$$\delta = c_3 \min\{\alpha_1, \eta_1\}\, \ell(Q), \tag{3.152}$$

where c_3 is a small parameter to be chosen soon. The rationale to choose c_3 is that we want $\phi(\widetilde{F})$ to lie in V_+^1, so that we may apply Lemma 3.44 and send $\phi(\widetilde{F})$ back into V_Q^1. Recall that V_+^1 is the set of points lying at most $\sigma/4$ far away from V_Q^1; recall also that $\widetilde{F} = \varphi_1(E_\rho \cap V^2) \subset V_Q^1$ by (3.122) (the way $\mathcal{F}$ was defined) and the property of $\{\varphi_t\}$, (3.127). Hence for an appropriate choice of c_3, say $c_3 = \frac{1}{300\sqrt{n}}$ (see (3.110)) we have that if $y \in W \cap \widetilde{F}$, then

$$\phi(y) \in V_+^1. \tag{3.153}$$

We set

$$F_1 := \phi(\widetilde{F}) \text{ and} \tag{3.154}$$

$$F_2 := h(F_1) = h(\phi(\widetilde{F})). \tag{3.155}$$

In particular, $F_2 \subset V_Q^1$ by Lemma 3.44.

9.2.2. *Construction of F_3.* We want to project F_2 back into a d-dimensional skeleton, since this is a requirement to belong to $\mathcal{F}_0$ (and so eventually to $\mathcal{F}$). By definition of $\mathcal{F}_0$, we will be projecting onto the d-skeleton of cubes coming from Δ_ρ.

We will use the following Lemma, which is taken from [**DS00**].

Lemma 3.45 (Lemma 11.14, [**DS00**].). *Let $j \in \mathbb{Z}$ and let A be a compact subset of $\mathbb{R}^n$ such that $\mathcal{H}^d(A) < \infty$. Denote by $N(A)$ the union of all the cubes $I \in \Delta_j$ that touch a cube in Δ_j which intersects A. Then there is a Lipschitz mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ with the following properties.*

$$f(x) = x \text{ for } x \in \mathbb{R}^n \setminus N(A); \tag{3.156}$$

$$f(x) = x \text{ for all } x \in \mathcal{S}_{j,d}; \tag{3.157}$$

$$f(A) \subset \mathcal{S}_{j,d}; \tag{3.158}$$

$$f(I) \subset I \text{ for all } I \in \Delta_j; \tag{3.159}$$

$$\mathcal{H}^d\left(f((A \cap I) \setminus \mathcal{S}_{j,d})\right) \le C\, \mathcal{H}^d\left((A \cap I) \setminus \mathcal{S}_{j,d}\right) \text{ for all } I \in \Delta_j. \tag{3.160}$$

Recall the definition of $\mathcal{S}_{j,d}$ in (3.16).

We now apply Lemma 3.45 with

$$j = -j(\rho) \text{ and } A = F_2 = h(\phi(\widetilde{F})),$$

and thus we set

$$F_3 = f(F_2) = (f \circ h \circ \phi)(\widetilde{F}). \tag{3.161}$$

Remark 3.46. Let us note a couple of facts. First, we see that if $y \in \widetilde{F} \setminus W$, then $\phi(y) = y$ (by definition of W, as in (3.141)); but, still with the same y, also $h(\phi(y)) = h(y) = y$, since $y \in V_Q^1$ already, and h does not move such points (as in (3.150)), and further, $f(h(\phi(y))) = f(y) = y$ by (3.157), since $y \in \widetilde{F}$, and therefore it belongs to the d-face of some cubes from Δ_ρ.

Lemma 3.47. *With the notation as above, we have*

$$\dim\left(F_3 \setminus \mathcal{S}_{-j(\rho),d}\right) \le d - 1. \tag{3.162}$$

Proof. By Remark 3.46, we already know that $f(h(\phi(\widetilde{F} \setminus W))) = \widetilde{F} \setminus W \subset \mathcal{S}_{-j(\rho),d}$. On the other hand, we must have that $f(h(\phi(\widetilde{F} \cap W))) \subset \mathcal{S}_{-j(\rho),d}$ by (3.158). Thus the Lemma follows. □

9.2.3. *Construction of F_4 and $F_4 \in \mathcal{F}$.* Note that F_3 is not necessarily a union of full d-dimensional faces: the projection f is into and not necessarily onto.

Lemma 3.48. *There exists a Lipschitz map $\pi : \mathbb{R}^n \to \mathbb{R}^n$ so that $\pi(F_3)$ is precisely the union of those d-dimensional faces which were contained in F_3.*

Proof. See [**Dav04**], pages 211-212. The idea of the proof is to consider those faces T which intersect F_3 on a set of positive d-dimensional measure but that are not contained in F_3. On these faces, exactly because they are not contained in F_3, we can define a Lipschitz map π which sends whatever lies of F_3 in one such face to its $(d-1)$-dimensional boundary. The result, F_4, will be a set of full d-faces plus a set of dimension smaller or equal to $d - 1$. □

We now set

$$F_4 := \pi(F_3). \tag{3.163}$$

LEMMA 3.49. *With notation as above,*

$$F_4 \in \mathcal{F}. \tag{3.164}$$

PROOF. Once again, see [**Dav04**], pages 212 to 215, from equation (5.32) to mid page 215. □

Hence F_4 is a valid competitor in $\mathcal{F}$ of $\widetilde{F}$. But $\widetilde{F}$ is a minimiser of the functional J in this class, and therefore we have the inequality

$$J(\widetilde{F}) \leq J(F_4). \tag{3.165}$$

We will use this inequalty in the following subsection to finally prove that $\widetilde{F}$ is also a quasi-minimiser of $\mathcal{H}^d$.

9.2.4. *$\widetilde{F}$ is a quasiminimiser.* First, note that $F_4 \subset F_3$, except perhaps from a set of dimension smaller than, or equal to, $d-1$. Thus, using also (3.165), we have that

$$J(\widetilde{F}) \leq J(F_3).$$

Recall the definition of W in (3.141) and that of J in (3.129). Writing $\widetilde{F} = \left(\widetilde{F} \cap W\right) \cup \left(\widetilde{F} \setminus W\right)$, and using the additivity of $\mathcal{H}^d$, we have

$$J(\widetilde{F}) = J(\widetilde{F} \cap W) + J(\widetilde{F} \setminus W). \tag{3.166}$$

Let us set

$$\Phi(y) = (f \circ h \circ \phi)\,(y).$$

With this notation we see that $F_3 = \Phi(\widetilde{F})$ (this is just (3.161)). Moreover, recall from Remark 3.46, that $\Phi(\widetilde{F}) = \Phi(\widetilde{F} \cap W) \cup \Phi(\widetilde{F} \setminus W)$, and in turn, that $\Phi(\widetilde{F} \setminus W) = \widetilde{F} \setminus W$, and thus $J(\Phi(\widetilde{F} \setminus W)) = J(\widetilde{F} \setminus W)$, which is immediate from the definition of J as in (3.129). In particular we get that

$$\begin{aligned} J(F_3) &= J(\Phi(\widetilde{F})) \\ &\leq J\left(\Phi(\widetilde{F} \cap W)\right) + J\left(\Phi(\widetilde{F} \setminus W)\right) \\ &= J\left(\Phi(\widetilde{F} \cap W)\right) + J\left(\widetilde{F} \setminus W\right). \end{aligned}$$

We thus have

$$J(\widetilde{F}) = J(\widetilde{F} \cap W) + J(\widetilde{F} \setminus W) \leq J(\Phi(\widetilde{F} \cap W)) + J(\widetilde{F} \setminus W),$$

which, subtracting $J(\widetilde{F} \setminus W)$ from both sides, gives,

$$J(\widetilde{F} \cap W) \leq J(\Phi(\widetilde{F} \cap W)).$$

But note that, by the definition of the functional J in (3.129),

$$\begin{aligned} &\mathcal{H}^d(\widetilde{F} \cap W) \leq J(\widetilde{F} \cap W) \leq J(\Phi(\widetilde{F} \cap W)) \\ &= \mathcal{H}^d(\Phi(\widetilde{F} \cap W) \cap E_\rho) + M\mathcal{H}^d(\Phi(\widetilde{F} \cap W) \setminus E_\rho) \leq M\mathcal{H}^d(\Phi(\widetilde{F} \cap W)). \end{aligned}$$

That is,

$$\mathcal{H}^d(\widetilde{F} \cap W) \leq M\,\mathcal{H}^d(\Phi(\widetilde{F} \cap W)). \tag{3.167}$$

Note that (3.167) resembles the comparison estimate (3.144): we need to swap Φ with ϕ. To do so, we need to show that up constants, the maps f and h did not increase the mass of $\phi(\widetilde{F} \cap W)$. Let us worry about f first. We write

$$A_1 := h(\phi(\widetilde{F} \cap W)) \cap \mathcal{S}_{-j(\rho),d}\,; \tag{3.168}$$

$$A_2 := h(\phi(\widetilde{F} \cap W)) \setminus \mathcal{S}_{-j(\rho),d}\,. \tag{3.169}$$

Now, because $f(y) = y$ whenever $y \in \mathcal{S}_{-j(\rho),d}$, we immediately have that

$$\mathcal{H}^d(f(A_1)) = \mathcal{H}^d(A_1). \tag{3.170}$$

Let us look at A_2. Because of (3.159) and the fact that dyadic cubes have bounded overlap, we can write

$$\mathcal{H}^d(f(A_2)) \leq \sum_{I\in\Delta_{-j(\rho)}} \mathcal{H}^d(f(A_2\cap I)) \leq C \sum_{I\in\Delta_{-j(\rho)}} \mathcal{H}^d(f(\overline{A_2}\cap I\setminus \mathcal{S}_{-j(\rho),d})).$$

To estimate this last sum, we use (3.160):

$$\begin{aligned} &C \sum_{I\in\Delta_{-j(\rho)}} \mathcal{H}^d(f(\overline{A_2}\cap I\setminus \mathcal{S}_{-j(\rho),d})) \\ &\leq C' \sum_{I\in\Delta_{-j(\rho)}} \mathcal{H}^d(\overline{A_2}\cap I\setminus \mathcal{S}_{-j(\rho),d}) \\ &\leq C'\mathcal{H}^d\left(\overline{h(\phi(\widetilde{F}\cap W))}\setminus \mathcal{S}_{-j(\rho),d}\right). \end{aligned}$$

Putting together these estimates, we see that

$$\mathcal{H}^d(\Phi(\widetilde{F}\cap W)) \leq \mathcal{H}^d(A_1) + C'\mathcal{H}^d(\overline{h(\phi(\widetilde{F}\cap W))}\setminus \mathcal{S}_{-j(\rho),d}). \tag{3.171}$$

LEMMA 3.50. *With the notation as above, we have*

$$\dim\left(\overline{h(\phi(\widetilde{F}\cap W))}\setminus (h(\phi(\widetilde{F}\cap W))\cup \mathcal{S}_{-j(\rho),d})\right) \leq d-1. \tag{3.172}$$

PROOF. This is equation 5.60 in [**Dav04**]. As the proof is brief, we add it for completeness. Set $X := h(\phi(\widetilde{F}\cap W))$. Let $x\in \overline{X}\setminus X$. Since W is compactly contained in B_2, and V_+^1 is closed, then there exists a $y\in \widetilde{F}\cap\overline{W}$ so that $x = h(\phi(y))$. Also, $y\notin W$, since $x\notin X$. But then, by definition of W in (3.141), we have that $\phi(y) = y$. Also, $y\in\widetilde{F}$, and $\widetilde{F}\subset V_Q^1\subset V_+^1$; hence $h(y) = y$. This implies that $x\in\widetilde{F}$. Now, if $x\in F^*$ (the coral part), then it belongs to $\mathcal{S}_{-j(\rho),d}$. But recall also from (3.121) that $\widetilde{F}$ is constituted by a subset of $\mathcal{S}_{-j(\rho),d}$ and a subset $L = \widetilde{F}\setminus\widetilde{F}^*$ of dimension lower than $d-1$. Thus we must have $\overline{X}\setminus X\cup(\mathcal{S}_{-j(\rho),d})\subset L$. This proves the lemma. □

Using (3.172), we then can write

$$\begin{aligned} &\mathcal{H}^d(A_1) + C'\mathcal{H}^d(\overline{h(\phi(\widetilde{F}\cap W))}\setminus \mathcal{S}_{-j(\rho),d}) \\ &\leq \mathcal{H}^d(A_1) + C'\mathcal{H}^d(h(\phi(\widetilde{F}\cap W))\setminus \mathcal{S}_{-j(\rho),d} \\ &= \mathcal{H}^d(A_1) + C'\mathcal{H}^d(A_2) \\ &\leq C'\mathcal{H}^d(h(\phi(\widetilde{F}\cap W))). \end{aligned} \tag{3.173}$$

Hence, (3.171) and (3.173) tell us that

$$\mathcal{H}^d(\Phi(\widetilde{F}\cap W)) \leq C'\mathcal{H}^d(h(\phi(\widetilde{F}\cap W))). \tag{3.174}$$

Now note that because π is Lipschtz with constant 4^n as for Lemma 3.44, we immediately see that

$$\mathcal{H}^d(h\circ\phi(\widetilde{F}\cap W)) \leq 4^{nd}\mathcal{H}^d(\phi(\widetilde{F}\cap W)). \tag{3.175}$$

The two estimates (3.174) and (3.175) together show that $\widetilde{F}\cap B_2$ is a (B_2,k,δ)-quasiminimal set (with B_2 as defined in (3.109)), $k = 4^{nd}C'M$ and

$$\delta = c_3\min\{\alpha_1,\eta_1\}\,\ell(Q).$$

This finishes the proof of Lemma 3.43.

9.3. A uniformly rectifiable set covering the minimising set $\widetilde{F}$. In this short subsection, we will use the main result of [**DS00**], to show that $\widetilde{F}$ can be locally covered by a uniformly rectifiable set.

LEMMA 3.51. *With notation as above (in particular recall the definition of B_1 in (3.109)), we have that*

$$\widetilde{F}\cap B_1\subset Z,$$

where Z is a uniformly d-rectifiable set.

Let us recall the main result in [DS00].

THEOREM 3.52 ([DS00], Theorem 2.11). *Let U be an open set in $\mathbb{R}^n$, and suppose that S is a (U,k,δ)-quasiminimizer for $\mathcal{H}^d$. Let S^* be the support in U of the restriction of $\mathcal{H}^d$ to S. Then for each $x \in S^*$ and radius R_0 which satisfy*

$$0 < R_0 < \delta \text{ and } B(x, 3R_0) \subset U, \tag{3.176}$$

there is a compact, Ahlfors d-regular set Z such that

$$S^* \cap B(x, R_0) \subset Z \subset S^* \cap B(x, 2R_0) \tag{3.177}$$

and

$$Z \text{ is uniformly rectifiable and contains big pieces of Lipschitz graphs.} \tag{3.178}$$

The constants for the Ahlfors regularity and for the uniform rectifiability of Z depend only on n and k.

PROOF OF LEMMA 3.51. Recall that $\widetilde{F}$ is a minimiser of the functional J over $\mathcal{F}$, see definitions (3.129) and (3.122). By Lemma 3.43, we know that $S = \widetilde{F} \cap B_2$ is a (B_2, k, δ)-quasiminimizer for $\mathcal{H}^d$. Also recall that

$$B_2 = B\left(x, r - \frac{8}{10}\eta_1 \ell(Q)\right) \text{ (see (3.109))},$$

that $k = C4^{nd}M$, and $\delta = c_3 \min\{\alpha_1, \eta_1\}\ell(Q)$. Then, making c_3 smaller if needed, we see that for all points $y \in \widetilde{F} \cap B_1$, we have

$$B(y, 3\delta) \cap \widetilde{F}^* \subset B_2, \tag{3.179}$$

where recall that $\widetilde{F}^*$ is the coral part of $\widetilde{F}$ (see (3.121)). By Theorem 3.52, we see that there exists a uniformly rectifiable set Z_y so that

$$\widetilde{F}^* \cap B(y, \delta/2) \subset Z_y \subset \widetilde{F}^* \cap B(y, \delta), \tag{3.180}$$

since we can chose $R_0 \geq \delta/2$. Now, clearly

$$\bigcup_{y \in B_1 \cap \widetilde{F}} B(y, \delta/2) \supset \widetilde{F} \cap B_1.$$

Moreover, we can find a finite subfamily of balls $\{B(y_j, \delta/2)\}_{j=0}^N$ such that

$$\widetilde{F} \cap B_1 \subset \bigcup_{i=0}^N B(y_i, \delta/2) \tag{3.181}$$

and

$$N \leq C = C(n, \eta_1, \alpha_1).$$

To see this, recall that $r(B_1) = r - 9/10\eta_1\ell(Q)$. This, by the choice of $r = r_2$ in (3.84), and choosing the constant C in (3.81) appropriately, gives that $r(B_1) \leq \ell(Q)/2$. Hence, since $\delta = c_3 \min\{\alpha_1, \eta_1\}\ell(Q)$, we need at most C balls in (3.181), where C depends only on n, α_1 and η_1. Now, for this each one of these balls, we take the corresponding uniformly rectifiable set Z_{y_i} as in (3.180), and we set

$$Z_x := \bigcup_{i=0}^N Z_{y_i}. \tag{3.182}$$

Then Z_x is a uniformly rectifiable set (with uniform constant now depending on N, and thus α_1 and η_1) so that

$$\widetilde{F}^* \cap B_1 \subset \bigcup_{i=0}^N B(y_i, \delta/2) \cap \widetilde{F}^* \subset \bigcup_{i=0}^N Z_{y_i} = Z_x.$$

This proves the lemma. □

REMARK 3.53. A short summary of what has been done so far will be useful to the reader in the subsequent section.

We started off with a surface E satisfying the topological condition (TC) with some prescribed parameters r_0, α_0, η_0 and δ_0. We took the corona construction from Lemma 2.4, and we showed that TC on E implies a skeletal topological condition on all the approximating E_R in the coronisation (Section 7, Lemma 3.30). Next, we constructed a further approximating Ahlfors regular set $E_{R,\rho}$ (see (3.104)), to then show that *for any point* $x \in R$ (see the choice of x_2 in (3.83)), there is a (B_2, k, δ)-quasiminimiser set $\widetilde{F} = \widetilde{F}(R,x)$ such that, first,

$$\mathcal{H}^d(\widetilde{F} \setminus E_{R,\rho}) \leq c_2 \delta_1 \ell(Q)^d, \tag{3.183}$$

— this is equation (3.135)); and second, that there exists a uniformly d-rectifiable set Z_x so that

$$B_1 \cap \widetilde{F}^* \subset Z_x; \tag{3.184}$$

— this is Lemma 3.51.

Lemma 3.54. *Let $R \in \mathsf{Top}(k_0)$. For each $Q \in \mathsf{Tree}(R)$, there exists a uniformly d-rectifiable set Z_Q and a set $\widetilde{F}_Q$ which is a the union of a uniformly finite family of quasiminimal sets so that,*

$$\mathcal{H}^d(\widetilde{F}_Q \cap E_{R,\rho}) \geq \delta_0 \ell(Q)^d, \tag{3.185}$$

and,

$$\mathcal{H}^d(\widetilde{F}_Q \setminus E_{R,\rho}) \leq c_2 \delta_1 \ell(Q)^d; \tag{3.186}$$

and,

$$\widetilde{F}_Q \subset Z_Q. \tag{3.187}$$

Proof. Now, recall from (3.109), that

$$B_1 = B\left(x_2, r_2 - \frac{9}{10}\eta_1 \ell(Q)\right),$$

and also recall from (3.51), that $r_2 > \eta_1 \ell(Q)$. In particular we have that

$$B(x, \eta_1 \ell(Q)/10) \subset B_1.$$

Note that to cover Q, we need at most $N' \sim_n \eta_1^{-n}$ balls centered on R and with radius $\eta_1 \ell(Q)/10$. Pick one such collection that is also minimal, and thus of bounded overlap. Let it be

$$B^j := B(x_j, \eta_1 \ell(Q)/10),\ x_j \in Q,\ 1 \leq j \leq N'.$$

For each $1 \leq j \leq N'$, there correspond a quasiminimal set $\widetilde{F} = \widetilde{F}(Q, x_j)$ (and its coral part $\widetilde{F}^*$) satisfying (3.183), and a corresponding uniformly d-rectifiable set Z_{x_j} satisfying (3.184). We now set

$$\widetilde{F}_Q := \bigcup_{j=1}^{N'} \widetilde{F}(Q, x_j), \tag{3.188}$$

and

$$Z_Q := \bigcup_{j=1}^{N'} Z_{x_j} \tag{3.189}$$

It is then easy to see that (3.185), (3.186) and (3.187) hold.

□

Lemma 3.55. *For each pair $(x,r) \in E_{R,\rho} \times (0, \ell(R))$, there exists a uniformly d-rectifiable set $Z_{x,r}$ such that*

$$\mathcal{H}^d(E_{R,\rho} \cap Z_{x,r} \cap B(x,r)) \geq \delta_0 r^d.$$

The Ahlfors regularity, uniform rectifiability and BPLG constants for the sets $Z_{x,r}$ are uniform in (x,r).

Proof. If $x \in E_{R,\rho}$, then by the construction of $E_{R,\rho}$ (as in (3.103) and (3.104)), there exists a dyadic cube $I \in \mathscr{C}_R$ such that $\operatorname{dist}(x, \partial_d I) < \ell(I)$. Recall also that for each $I \in \mathscr{C}_R$, there exists a surface cube $Q_I \in \mathsf{Tree}(R)$ such that $\mathrm{dist}(Q_I, I) \leq c\tau^{-1}\ell(I)$ and $\ell(I) \leq \ell(Q_I) \leq c'\tau^{-1}\ell(I)$. This cube is given by Lemma 3.22.

Now, take two constants C, C' to be fixed below, depending on the constants c, c'.

(1) Suppose first that
$$C\tau^{-1}\ell(I) \leq r \leq C'\ell(Q_I).$$
Choosing C appropriately, we can insure that
$$B(x,r) \supset B_{Q_I}.$$
But from Lemma 3.54, we know that
$$\mathcal{H}^d(E_{R,\rho} \cap Z_{Q_I}) = \mathcal{H}^d(E_{R,\rho} \cap Z_{Q_I} \cap B(x,r)) \geq \delta_1 \ell(Q)^d.$$
Since $r \sim \ell(Q_I)$, where the constants behind $\sim$ depend on C, C', c, c', then we conclude that there is an absolute constant C'' so that
$$\mathcal{H}^d(E_{R,\rho} \cap Z_{Q_I} \cap B(x,r)) \geq C''\delta_0 r^d.$$
This give the Lemma for this case.

(2) Suppose now that
$$0 < r < C\tau^{-1}\ell(I).$$
Let T be a d-face of $\partial_d I$, and let $T(r)$ be tile of T containing x and with
$$\ell(T(r)) \sim \min\left\{\frac{1}{10}r, \ell(I)\right\}.$$
Then clearly,
$$T(r) \subset E_{R,\rho} \cap B(x,r) \text{ for all } r > 0,$$
and $T(r)$ is a uniform rectifiable set with constants independent of r. Now, note that if $C\tau^{-1} > r > \ell(I)$, then
$$\mathcal{H}^d(E_{R,\rho} \cap T(r) \cap B(x,r)) \geq \mathcal{H}^d(T(r)) = c\ell(I)^d \sim_{\tau,C,C'} r^d.$$
On the other hand, if $0 < r < \ell(I)$, we have
$$\mathcal{H}^d(E_{R,\rho} \cap B(x,r) \cap T(r)) \geq \mathcal{H}^d(T(r)) = cr^d.$$
In any case, we found a unifromly rectifiable set which intersects $E_{R,\rho}$ with measure bounded below uniformly. This gives the Lemma in this case.

(3) Now, if $C'\ell(Q_I) < r < \ell(R)$, we can repeat the arguments of point (1) for some parent of Q_I appropriately chosen. By construction, this parent will be belong to $\mathsf{Tree}(R)$, and thus the same estimates apply.

□

From the theory of uniformly rectifiable sets (see [**DS93**]), we deduce the following.

Corollary 3.56. *With notation as above, $E_{R,\rho} = E_{R,\rho}$ is a uniformly d-rectifiable set with Ahlfors regularity, uniform rectifiability and BPLG constants dependent only on those of the intersecting UR sets $Z_{x,r}$.*

For future use, let us pin down an easy fact about the distance between $R \subset E$ and $E_\rho = E_{R,\rho}$.

Lemma 3.57. *For each $x \in S$, with $S \in \mathsf{Stop}(R)$, we have*
$$\mathrm{dist}(x, E_{R,\rho}) \leq C\ell(S).$$

Proof. Using again Lemma 3.21, we see that if $S \in \mathsf{Stop}(R)$, then there exists a cube $I_S \in \mathscr{C}_R$ such that $\zeta_S \in I_S$ and $\ell(S) \sim \tau^{-1}\ell(I_S)$. Thus in particula, if $x \in S$, then $\mathrm{dist}(x, E_R) \lesssim \ell(S) \sim \tau^{-1}\ell(I_S)$. Further, by construction of $E_{R,\rho} = E_{R,\rho}$, we have that $E_R \subset E_{R,\rho}$. This proves the Lemma. □

10. Estimates on the β coefficients and the end of the proof

In this section we give the final estimates on the Jones' β coefficient which will prove Theorem 3.5.

Recall the notation $\mathcal{D}(k_0)$ from Lemma 2.4. Theorem 3.5 will easily follow from the Lemma below.

LEMMA 3.58. *Let Q_0, p and the parameters $r_0, \alpha_0, \delta_0, \eta_0$ of the topological condition be as in Theorem 3.5. Fix an arbitrary (but sufficiently large) integer $k_0 > 0$. Then if $\mathsf{A} \geq 1$ we have, with the above notation,*

$$\sum_{\substack{Q \subset Q_0 \\ Q \in \mathcal{D}(k_0)}} \beta_E^{p,d}(\mathsf{A}\, Q)^2 \ell(Q)^d \leq C\mathcal{H}^d(E \cap B_{Q_0}), \tag{3.190}$$

where C depends on n, d, A, but not on k_0.

We will use the following lemma.

LEMMA 3.59 ([AS1], Lemma 2.21). *Let $1 \leq p < \infty$ and $E_1, E_2 \subset \mathbb{R}^n$. Let $x \in E_1$ and fix $r > 0$. Take some $y \in E_2$ so that $B(x,t) \subset B(y, 2t)$. Assume that E_1, E_2 are both lower content d-regular. Then*

$$\beta_{E_1}^{p,d}(x,t) \lesssim_c \beta_{E_2}^{p,d}(y, 2t) + \left(\frac{1}{t^d} \int_{E_1 \cap B(x, 2t)} \left(\frac{\operatorname{dist}(y, E_2)}{t} \right)^p d\mathcal{H}^d_\infty(y) \right)^{\frac{1}{p}}.$$

Let us now get started. Recall that the definition of $\mathsf{Forest}(R)$ in (3.35).

SUBLEMMA 3.60. *Let $p = p(d)$ be as in (3.11) and fix an arbitrary integer $k_0 > 0$. Let $R \in \mathsf{Top}$. Then if $\mathsf{A} \geq 1$, with the above notation we have*

$$\sum_{\substack{Q \in \mathsf{Tree}(R) \\ Q \in \mathcal{D}(k_0)}} \beta_E^{p,d}(\mathsf{A}\, Q)^2 \ell(Q)^d \lesssim \ell(R)^d, \tag{3.191}$$

where the constant behind the symbol $\lesssim$ depends on n, d, A, and the Ahlfors regularity constant of E_R, but not on k_0.

PROOF. We want to apply Lemma 3.59 with $E_1 = E$ and $E_2 = E_{R,\rho}$. For $Q \in \mathcal{D}$, recall that ζ_Q denotes the center of Q. By the definition of $\mathsf{Tree}(R)$, we see that if $Q \in \mathsf{Tree}(R)$, then there must exists a dyadic cube $I \in \mathscr{C}_R$ which meets Q. The d-skeleton $\partial_d I$ of I is part of $E_{R,\rho}$. We see that $\ell(I) \lesssim \tau \ell(Q)$. Hence there exists a point $x'_Q \in E_{R,\rho}$ such that $|x_Q - x'_Q| \leq 4\tau\ell(Q)$, and we obtain that

$$B_Q := B(x_Q, \ell(Q)) \subset B(x'_Q, 2\ell(Q)) =: B'_Q.$$

This implies that for each cube $Q \in \mathsf{Tree}(R)$ the hypotheses of Lemma 3.59 are satisfied (with $E_1 = E$ and $E_2 = E_{R,\rho}$); we may then write

$$\begin{aligned} \sum_{\substack{P \in \mathsf{Tree}(R) \\ P \in \mathcal{D}(k_0)}} \beta_E^{p,d}(\mathsf{A}\, B_P)^2 \ell(P)^d &\lesssim \sum_{\substack{P \in \mathsf{Tree}(R) \\ P \in \mathcal{D}(k_0)}} \beta_{E_{R,\rho}}^{p,d}(2\mathsf{A}\, B'_P)^2\, \ell(P)^d \\ &+ \sum_{\substack{P \in \mathsf{Tree}(R) \\ P \in \mathcal{D}(k_0)}} \left(\frac{1}{\ell(P)^d} \int_{2\mathsf{A}\, B_P \cap E} \left(\frac{\operatorname{dist}(y, E_{R,\rho})}{\ell(P)} \right)^p d\mathcal{H}^d_\infty(y) \right)^{\frac{2}{p}} \ell(P)^d \\ &:= I_1 + I_2. \end{aligned}$$

First, let us look at I_1. We apply Theorem 5.2 to $E_{R,\rho}$; let us denote the cubes so obtained by $\mathcal{D}_{E_{R,\rho}}$. Note that for each $P \in \mathsf{Tree}(R)$ with $P \in \mathcal{D}(k_0)$, x'_P belongs to some cube $P' \in \mathcal{D}_{E_{R,\rho}}$ so that $\ell(P') \sim \ell(P)$; hence there exists a constant $C_4 \geq 1$ so that

$$2\mathsf{A}B'_P \subset C_4 B_{P'}.$$

This in turn implies that $\beta^{p,d}_{E_{R,\rho}}(2\mathsf{A}B'_P) \lesssim_{p,n,d,\mathsf{A},C_4} \beta^{p,d}_{E_{R,\rho}}(C_4 B_{P'})$. Hence,

$$\sum_{\substack{P\in\mathsf{Tree}(R)\\ P\in\mathcal{D}(k_0)}} \beta^{p,d}_{E_{R,\rho}}(2\mathsf{A}B'_P)^2\ell(P)^d \lesssim_{p,n,d,\mathsf{A},C_1} \sum_{\substack{P'\in\mathcal{D}_{E_{R,\rho}}\\ \ell(P')\lesssim\ell(R)}} \beta^{p,d}_{E_{R,\rho}}(C_4 B_{P'})^2\ell(P')^d. \tag{3.192}$$

Since $E_{R,\rho}$ is uniformly rectifiable (by Corollary 3.56), we immediately have that $I_1 \lesssim \ell(R)^d$ by Theorem 1.2. Let us remark that the content β number we are using are comparable to the one introduced by David and Semmes when computed on Ahlfors regular sets.

We now now estimate I_2. Let $y \in 2\mathsf{A}R$; by Lemma 3.57, there exists a cube $S \in \mathsf{Stop}(R)$ such that

$$\operatorname{dist}(y, E_{R,\rho}) \lesssim \ell(S); \tag{3.193}$$

We can estimate the integral in I_2 with (3.193) as follows.

$$\begin{aligned}\int_{2\mathsf{A}B_P\cap E}\left(\frac{\operatorname{dist}(y,E_{R,\rho})}{\ell(P)}\right)^p d\mathcal{H}^d_\infty(y) &\leq \sum_{P'\in\mathcal{N}(P)}\int_{P'}\left(\frac{\operatorname{dist}(y,E_{R,\rho})}{\ell(P)}\right)^p d\mathcal{H}^d_\infty(y)\\ &\lesssim \sum_{P'\in\mathcal{N}(P)}\sum_{\substack{S\in\mathsf{Stop}(R)\\ S\subset P'}}\int_S\frac{\ell(S)^p}{\ell(P)^p}\\ &\lesssim \sum_{P'\in\mathcal{N}(P)}\sum_{\substack{S\in\mathsf{Stop}(R)\\ S\subset P'}}\frac{\ell(S)^{d+p}}{\ell(P)^p},\end{aligned}$$

and so

$$I_2 \lesssim \sum_{\substack{P\in\mathsf{Tree}(R)\\ P\in\mathcal{D}(k_0)}}\sum_{P'\in\mathcal{N}(P)}\sum_{\substack{S\in\mathsf{Stop}(R)\\ S\subset P'}}\frac{\ell(S)^{\frac{2d}{p}+2}}{\ell(P)^{d(\frac{2}{p}-1)+2}}.$$

We now swap the sums (which are all finite), to obtain that

$$\begin{aligned}I_2 &\lesssim \sum_{S\in\mathsf{Stop}(R)}\ell(S)^{\frac{2d}{p}+2}\sum_{\substack{P\in\mathsf{Tree}(R)\\ \exists P'\in\mathcal{N}(P)\colon P'\supset S}}\frac{1}{\ell(P)^{d(\frac{2}{p}-1)+2}}\\ &\lesssim_{d,n} \sum_{S\in\mathsf{Stop}(R)}\ell(S)^{\frac{2d}{p}+2}\sum_{\substack{P\in\mathsf{Tree}(R)\\ \exists P'\in\mathcal{N}(P)\colon P'\supset Q}}\frac{1}{\ell(P)^{d(\frac{2}{p}-1)+2}}.\end{aligned} \tag{3.194}$$

We see that the number of cubes $P \in \mathsf{Tree}(R)$ of a given generation so that there exists a sibling $P' \in \mathcal{N}(P)$ for which $P' \supset S$ is bounded above by a universal constant depending on n and A. Thus we can sum geometrically the interior series:

$$\sum_{\substack{P\in\mathsf{Tree}(R)\\ \exists P'\in\mathcal{N}(P)\colon P'\supset Q}}\frac{1}{\ell(P)^{d(\frac{2}{p}-1)+2}} \lesssim_n \frac{1}{\ell(S)^{d(\frac{2}{p}-1)+2}}.$$

Therefore we obtain

$$(3.194) \lesssim \sum_{S\in\mathsf{Stop}(R)}\frac{\ell(S)^{d(\frac{2}{p})+2}}{\ell(S)^{d(\frac{2}{p}-1)+2}} = \sum_{S\in\mathsf{Stop}(R)}\ell(S)^d.$$

This latter sum is bounded above by $C\ell(R)^d$. This concludes the proof of the lemma. □

Proof of Lemma 3.58. With Q_0 as in the statement of the theorem, we write

$$\sum_{\substack{Q\subset Q_0\\ Q\in\mathcal{D}(k_0)}}\beta^{p,d}_E(\mathsf{A}Q)^2\ell(Q)^d \lesssim \sum_{\substack{R\in\mathsf{Top}\\ R\subset Q_0}}\sum_{\substack{Q\in\mathsf{Tree}(R)\\ Q\in\mathcal{D}(k_0)}}\beta^{p,d}_E(\mathsf{A}Q)^2\ell(Q)^d. \tag{3.195}$$

By Lemma 3.60, we see that

$$(3.195) \lesssim \sum_{\substack{R \in \mathsf{Top} \\ R \subset Q_0}} \ell(R)^d. \tag{3.196}$$

Note that each $R \in \mathsf{Top}$ is the child of some stopped cube R'. Recall we stopped at a surface cube $R' \in \mathcal{D}$ whenever it happened that $R' \cap I$ and $\ell(I) \sim \ell(Q)$ for some $I \in \mathsf{Bad}$. We can therefore associate to each $R \in \mathsf{Top}$ a bad dyadic cube I, and thus, by (2.17), we have that

$$\sum_{\substack{R \in \mathsf{Top} \\ R \subset Q_0}} \ell(R)^d \lesssim \sum_{\substack{I \in \mathsf{Bad} \\ I \subset B_{Q_0}}} \ell(I)^d \lesssim \mathcal{H}^d(E \cap B_{Q_0}). \tag{3.197}$$

The estimate in (2.17) is independent of k_0, so is the one we obtained here. All in all, we see that,

$$(3.195) \lesssim \mathcal{H}^d(E \cap B_{Q_0}).$$

This concludes the proof of Lemma 3.58. □

Proof of Theorem 3.5. With Lemma 3.58, the theorem follows immediately by taking $k_0 \to \infty$ and recalling that the estimate (3.190) is independent of k_0. □

Proof of Corollary 3.7. It follows from [**AS18**, equation 11.2], which is a 'cubes version' of Theorem I in [**AS18**] that

$$\mathcal{H}^d(Q_0) \lesssim \ell(Q_0)^d + \sum_{\substack{Q \in \mathcal{D} \\ Q \subset Q_0}} \beta_E^{d,p}(\mathsf{A}Q)^2 \operatorname{diam}(Q)^d \tag{3.198}$$

whenever A is sufficiently large. This completes the proof of Corollary 3.7. □

11. An application to uniformly non-flat sets

In [**Dav04**], David proved that if E is a topologically stable d-surface and it is uniformly non-flat, then it must have dimension strictly larger that d. As mentioned in the introduction, David's result was in the spirit of a previous result by Bishop and Jones about uniformly wiggly, or uniformly non-flat, sets.

Definition 3.61. A set $E \subset \mathbb{R}^n$ is called *uniformly wiggly* or *uniformly non-flat* (with parameter β_0) if for all cubes $Q \in \mathcal{D}_E$, we have that

$$\beta_\infty(Q) > \beta_0 > 0.$$

Remark 3.62. Clearly, this definition can be recast in terms of different types of β numbers, such as the content beta numbers which we have been using so far in this paper.

Let us now recall the result of Bishop and Jones.

Theorem 3.63 ([**BJ97**], Theorem 1.1). *Let $E \subset \mathbb{R}^2$ be a compact, connected subset which is uniformly wiggly with parameter β_0. Then $\dim(E) > 1 + C\beta_0^2$, where C is an absolute constant*[4].

Let us go back to David's result. His is, in a sense, a generalisation of Bishop and Jones's Theorem. However, it is of qualitative nature, and the dependence of the lower bound on the parameter β_0 is not explicit. In this section we give a generalisation of Bishop and Jones Theorem where such a dependence is made explicit. This result is a fairly immediate application of Corollary 3.7 and of the scheme of proof from [**BJ97**].

Theorem 3.64. *Let $E \subset \mathbb{R}^n$ be a topologically stable d-surface. Let $R \in \mathcal{D}$ be such that, for any $Q \in \mathcal{D}(R)$, we have that*

$$\beta_E^{p,d}(C_0 Q)^2 > \beta_0 > 0. \tag{3.199}$$

Then

$$\dim(R) > d + c\beta_0^2. \tag{3.200}$$

[4]We remark once more that by dimension we mean Hausdorff dimension. See [**Mat95**], Definition 4.8 for a definition.

The scheme of the proof is the same as that of Bishop and Jones. We also used a clear summary of such proof to be found in Garnett and Marshall's book, [GM], page 429. For this reason, we only sketch the proof.

Proof. Given a TS d-surface, a cube $R \in \mathcal{D}(E)$ and an integer $m \geq 0$, we put

$$\beta_m(R) = \sum_{Q \in \mathcal{D}_m(R)} \beta_E^{p,d}(Q)\ell(Q)^d.$$

Next, we consider

$$\Delta_{k,c}(R) := \left\{ I \in \Delta \,|\, I \cap R \neq \emptyset \text{ and } \ell(I) = c\,2^{-k} \right\}, \tag{3.201}$$

where $c < 1$ is a constant which is a power of 2 and will be fixed later (it will depend on the parameter $\lambda > 0$ coming from Theorem 5.2). We then put

$$E_{R,k} := \bigcup_{I \in \Delta_{k,c}(R)} \partial_d I.$$

Claim 1. There exists a constant C_5 so that, if

$$R \in \mathcal{D}_{N_0}(E) \tag{3.202}$$

with $N_0 \leq k$, then

$$C_5 \left(\ell(R)^d + \sum_{m=N_0}^{k} \beta_m(R) \right) \leq \mathcal{H}^d(E_{R,k}). \tag{3.203}$$

To see this, note first that because E satisfies the topological condition (TC) with parameters $r_0, \alpha_0, \eta_0, \delta_0$, then $E_{R,k}$ must also be a TS d-surface with comparable parameters (up to constants). Hence, we can apply Corollary 3.7 to see that

$$\mathcal{H}^d(E_{R,k}) \sim \beta_{E_{R,k},C_0,p}(R),$$

where the constants behind $\sim$ are as in the statement of Corollary 3.7.

We can now check (3.203): we have that

$$\mathcal{H}^d(E_{R,k}) \sim \operatorname{diam}(E_{R,k})^d + \sum_{P \in \mathcal{D}_{E_{R,k}}} \beta_{E_{R,k}}(C_0 P)^2\, \ell(P)^d$$

By construction, we immediately see that $\operatorname{diam}(E_{R,k})^d \sim \ell(R)^d$. On the other hand, consider a cube $Q \in \mathcal{D}_E$, such that $\ell(Q) > c2^{-k}$, for $c < 1$ as in (3.201). If we choose c sufficiently small, we can apply Lemma 3.59 with $E_1 = E$ and $E_2 = E_{R,k}$, to obtain

$$\beta_E^{p,d}(C_0 P) \lesssim \beta_{E_{R,k}}^{p,d}(2C_0 P) + \left(\frac{1}{\ell(P)^d} \int_{2C_0 B_P} \left(\frac{\operatorname{dist}(y, E_{R,k})}{\ell(P)} \right)^p d\mathcal{H}^d_\infty \right)^{\frac{1}{p}}.$$

Thus we see that

$$\sum_{\substack{P \in \mathcal{D}_E \\ \ell(P) > c2^{-k}}} \beta_E^{p,d}(C_0 P)$$
$$\lesssim \sum_{\substack{P' \in \mathcal{D}_{E_{R,k}} \\ \ell(P') \gtrsim c2^{-k}}} \beta_{E_{R,k}}^{p,d}(2C_0 P') + \sum_{\substack{P \in \mathcal{D}_E \\ \ell(P) > c2^{-k}}} \left(\frac{1}{\ell(P)^d} \int_{2C_0 B_P} \left(\frac{\operatorname{dist}(y, E_{R,k})}{\ell(P)} \right)^p \mathcal{H}^d_\infty(y) \right)^{\frac{1}{p}}.$$

With a calculation similar to that in Sublemma 3.60, we obtain that the second sum above is $\lesssim \ell(R)^d$. This then gives

$$\begin{aligned}\mathcal{H}^d(E_{R,k}) &\sim \ell(R)^d + \sum_{P\in\mathcal{D}_{E_{R,k}}} \beta^{p,d}_{E_{R,k}}(C_0P)^2\ell(P)^d \\ &\gtrsim \ell(R)^d + \sum_{\substack{P'\in\mathcal{D}_{E_{R,k}} \\ \ell(P')\geq c2^{-k}}} \beta^{p,d}_{E_{R,k}} + \sum_{\substack{P\in\mathcal{D}_E \\ \ell(P)>c2^{-k}}} \left(\frac{1}{\ell(P)^d}\int_{2C_0B_P}\left(\frac{\operatorname{dist}(y,E_{R,k})}{\ell(P)}\right)^p \mathcal{H}^d_\infty(y)\right)^{\frac{1}{p}} \\ &\geq C_5\left(\ell(R)^d + \sum_{\substack{P\in\mathcal{D}_E \\ \ell(P)\geq c2^{-k}}} \beta^{p,d}_E(C_0P)\right).\end{aligned}$$

This proves (3.203).

Claim 2. Let N an integer so that $N > N_0$ (recall that N_0 is roughly the scale of R, see (3.202)). Consider a dyadic cube $I_N \in \Delta_N(\mathbb{R}^n)$ for which $\ell(I_N) < \ell(R)/10$ and such that $\frac{1}{3}I_N \cap E \neq \emptyset$. For $k > N$, we have

$$\sum_{m=N}^{k} \beta_m(R\cap I_N)^2 \geq (k-N)\beta_0^2 2^{-dN}.$$

By $\beta_m(R\cap I_N)$ here we mean that we sum over those cubes $Q \in \mathcal{D}_m(R)$ such that $Q\cap I_N \neq \emptyset$. To see this, note first that by lower regularity of E, there are at least $2^{d(m-N)}$ (up to a a constant depending on the lower regularity parameter) dyadic cubes J of generation m (with $m > N$) such that $J \subset I_N$ and $J \cap E \neq \emptyset$. Hence since E is uniformly non-flat, we see that if $N \leq m \leq k$,

$$\begin{aligned}\beta_m(R\cap I_N)^2 &= \sum_{\substack{Q\in\mathcal{D}_m(R) \\ Q\cap I_N\neq\emptyset}} \beta_E(C_0Q)^2\ell(Q)^d \\ &\geq \beta_0^2 \sum_{\substack{Q\in\mathcal{D}_m(R) \\ Q\cap I_N}} \ell(Q)^d \\ &\sim \beta_0^2 \sum_{\substack{J\in\Delta_{m,c}(R) \\ J\subset I_N}} \ell(J)^d \\ &\gtrsim_c \beta_0^2 2^{d(m-N)}2^{-dm} \sim_c \beta_0^2 2^{-dN}.\end{aligned}$$

Hence, we have that

$$\sum_{m=N}^{k} \beta_m(R\cap I_N)^2 \gtrsim_c (k-N)\beta_0^2\, 2^{-dN},$$

and so, using (3.203),

$$\mathcal{H}^d(E_{R,k}\cap I_N) \gtrsim_{C_5,c} (k-N)\,\beta_0^2\, 2^{-dN}.$$

Let now $\{z_j\}$, j in some index set A, be a maximal 2^{-k}-separated net of $E_{R,k}\cap I_N$ such that

$$\bigcup_{j\in A} B(z_j, 2^{-k+2}) \supset E_{R,k}\cap I_N. \tag{3.204}$$

Then there exists a constant C_7 (depending only on n) so that

$$\mathcal{H}^d(E_{R,k}\cap I_N) \leq C_7\, c^d 2^{-dk}\operatorname{Card}(A).$$

Thus we obtain

$$C_7\, c^d 2^{-dk}\operatorname{Card}(A) \gtrsim_{C_5,c} (k-N)\,\beta_0^2 2^{-dN},$$

and therefore

$$\operatorname{Card}(A) \gtrsim_{C_5,C_7,c} (k-N)\,\beta_0^2\, 2^{d(k-N)}. \tag{3.205}$$

Since k was an arbitrary integer with $k \geq N$, we can choose it so that
$$\kappa := k - N \sim \frac{1}{\beta_0^2}.$$
Hence we see from (3.205) that

$$\operatorname{Card}(A) \geq 2^{(d+c'\beta_0^2)\kappa}, \tag{3.206}$$

where $c' = c'(C_7, C_5, c)$.

We now apply this construction recursively for each $N > N_0$, as follows. For N_0, we put
$$\mathcal{S}_0 := \{I \in \Delta_{N_0+\kappa}(R) \,|\, \exists j \in A \text{ s.t. } z_j \in I\}$$
Then for each $I \in \mathcal{S}_0$, we find a maximal net $\{z_j\}_{j\in A}$ as in (3.204); the cardinality of this net will be again as in (3.206). We put the relative cubes in the subfamily
$$\mathcal{S}(I) := \{J \in \Delta_{N_0+2\kappa} \,|\, \exists j \in A \text{ s.t. } z_j \in J\}\,.$$
We then put
$$\mathcal{S}_1 := \bigcup_{I\in\mathcal{S}_0} \mathcal{S}(I).$$
Having defined $\mathcal{S}_{j-1}$, we set
$$\mathcal{S}_j := \bigcup_{I\in\mathcal{S}_{j-1}} \mathcal{S}(I),$$
where $\mathcal{S}(I) = \{j \in \Delta_{N_0+j\kappa} \,|\, \exists j \in A \text{ s.t. } z_j \in J\}$. Let us record that for each $j \in \mathbb{N}$, we have

(1) Each $J \in \mathcal{S}_j$, is a subset of some $I \in \mathcal{S}_{j-1}$.
(2) Each $I \in \mathcal{S}_{j-1}$ contains at least $2^{(d+c'\beta_0^2)\kappa}$ cubes $I \in \mathcal{S}_j$ (as in (3.205)).
(3) For each $j \in \mathbb{N}$, if $I \in \mathcal{S}_j$, we have $I \cap R \neq \emptyset$.

Claim 3. If R satisfies (1)-(3), then
$$\dim(R) > d + c'\beta_0^2.$$
To prove this claim, we define the μ on the elements I of $\mathcal{S}_j$, for $j \geq 0$, by
$$\mu(I) = \operatorname{Card}(A)^{-j} \leq 2^{-j\kappa(d+c'\beta_0^2)}.$$
One can then check that $\operatorname{spt}(\mu) = E$ and that $\mu(R) = 1$. Then, by Frostman's Lemma (Theorem 8.8 in [Mat95]), we have that
$$\mathcal{H}^{d+c'\beta_0^2}(R) > 0.$$
This completes the proof of Theorem 3.64. □

12. Appendix to Chapter 3

Recall the statement of Lemma 3.21.

LEMMA. *Let S be a cube in $\mathsf{Stop}(Q)$ for some $Q \in \mathsf{N}ext(R)$, $R \in \mathsf{Top}(k_0)$. Then there exists a dyadic cube $I_S := I \in \mathscr{C}_Q$ so that $I_S \subset \frac{1}{2}B_S$ and $\ell(I_S) \sim \tau\ell(S)$.*

PROOF OF LEMMA 3.21. Let z_S be the center of S. Then there exists a dyadic cube $I \in \mathscr{C}_Q$ such that $z_S \in I$; thus for I we have $\operatorname{dist}(I,S) = 0$, and therefore $d_Q(I) \leq \operatorname{dist}(I,S) + \ell(S) = \ell(S)$. In other words, when computing $d_Q(I)$, it suffices to minimise over all cubes T such that
$$\operatorname{dist}(I,T) + \ell(T) \leq \ell(Q).$$
But note that since S is a minimal cube in $\mathsf{Tree}(Q)$, we must have that $T \subset S^c$. Recall also that, by Theorem 5.2, $E \cap B(z_S, c_0\ell(S)) \subset S$. If we let τ be small enough, we can insure that $I \subset B(z_S, \frac{c_0\ell(S)}{2})$; hence we see that
$$\operatorname{dist}(I,T) \gtrsim \ell(S),$$
and therefore $\tau^{-1}\ell(I) \sim d_Q(I) \gtrsim \ell(S) \geq \ell(I)$ □

LEMMA. *Let* $I \in \mathscr{C}_Q$ *for* $Q \in \mathsf{Next}(R)$, $R \in \mathsf{Top}(k_0)$. *Then there exists a cube* $Q_I \in \mathsf{Tree}(Q)$ *so that*

$$\ell(I) \le \ell(Q_I) \le c\tau^{-1}\ell(I); \tag{3.207}$$

$$\mathrm{dist}(I, Q_I) \le c\tau^{-1}\ell(I). \tag{3.208}$$

PROOF OF LEMMA 3.22. Recall that $d_Q(I)\tau \sim \ell(I)$. Now, by definition of $d_Q(I)$, there exists a cube $Q' \in \mathsf{Stop}(Q)$ such that $\mathrm{dist}(I, Q') + \ell(Q) \le 1.5 d_Q(I) \sim \tau^{-1}\ell(I)$. This immediately implies (3.208) and the second inequality in (3.207). As for the first one, if it doesn't hold, it suffices to take some ancestor of Q' in $\mathsf{Tree}(Q)$. We then let this ancestor to be Q_I. $\square$

Part 2

A conjecture of Carleson

CHAPTER 4

A proof of Carleson ϵ^2 conjecture

1. Introduction and statement of the results

This chapter is dedicated to provide a positive resolution to a longstanding conjecture of L. Carleson.

We described the conjecture in the introduction. We recall here the problem and all the relevant notation. Let Ω^+ be a proper open set in $\mathbb{R}^2$, and set $\Gamma = \partial\Omega^+$ and $\Omega^- = \mathbb{R}^2 \setminus \overline{\Omega^+}$. For $x \in \mathbb{R}^2$ and $r > 0$, denote by $I^+(x,r)$ and $I^-(x,r)$ the longest open arcs of the circumference $\partial B(x,r)$ contained in Ω^+ and Ω^-, respectively (they may be empty). Then we define

$$\epsilon(x,r) = \frac{1}{r} \max\left(\left|\pi r - \mathcal{H}^1(I^+(x,r))\right|, \left|\pi r - \mathcal{H}^1(I^-(x,r))\right|\right).$$

Here $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. The Carleson ϵ^2-square function is given by

$$\mathcal{E}(x)^2 := \int_0^1 \epsilon(x,r)^2 \, \frac{dr}{r}. \tag{4.1}$$

If Γ is a line then $\mathcal{E}(x) = 0$ for all $x \in \Gamma$. Carleson conjectured that (4.1) encodes some regularity properties of Γ.

Conjecture 4.1 (Carleson's ϵ^2-conjecture). *Suppose Γ is a Jordan curve. Except for a set of zero $\mathcal{H}^1$-measure, Γ has a tangent at $x \in \Gamma$ if and only if $\mathcal{E}(x) < \infty$.*

See Section 2.5 below for the precise definition of a tangent. This conjecture may be found in Bishop [**B**, Conjecture 3], Bishop-Jones [**BJ94**], Garnett-Marshall [**GM**, p.220], and David-Semmes [**DS91**, Section 21], for example.

The 'if' direction of Conjecture 4.1 is a well-known result: When Γ is a Jordan curve, it follows from the Ahlfors distortion theorem and an argument of Beurling that

$$\mathcal{E}(x) < \infty \qquad \text{for } \mathcal{H}^1\text{-a.e. tangent point of } \Gamma.$$

See [**BJ94**, p.79], for example. Therefore, the content of Conjecture 4.1 is that the converse statement should also hold true.

Thus the aim of this chapter is to prove Conjecture 4.1. This proof is the one that appeared in [**JTV**]. We also show that an analogue of this result also holds for two-sided corkscrew open sets, which have a scale invariant topological assumption but are not necessarily connected, see Section 2.4 below for the definition. Our precise result is the following theorem.

Theorem 4.2. *Let $\Omega^+ \subset \mathbb{R}^2$ be either a Jordan domain or a two-sided corkscrew open set, let $\Gamma = \partial\Omega^+$, and let $\mathcal{E}$ be the associated square function defined in (4.1). Then the set $G = \{x \in \Gamma : \mathcal{E}(x) < \infty\}$ is rectifiable and at $\mathcal{H}^1$-almost every point of G there exists a tangent to Γ.*

Recalling the definition of rectifiable sets in (1.1), Chapter 1, we obtain the following corollary.

Corollary 4.3. *Let $\Gamma \subset \mathbb{R}^2$ be a Jordan curve, and let $\mathcal{E}$ be the associated square function defined in (4.1). Then, the set of tangent points of Γ coincides with the points $x \in \Gamma$ such that $\mathcal{E}(x) < \infty$, up to a set of zero measure $\mathcal{H}^1$.*

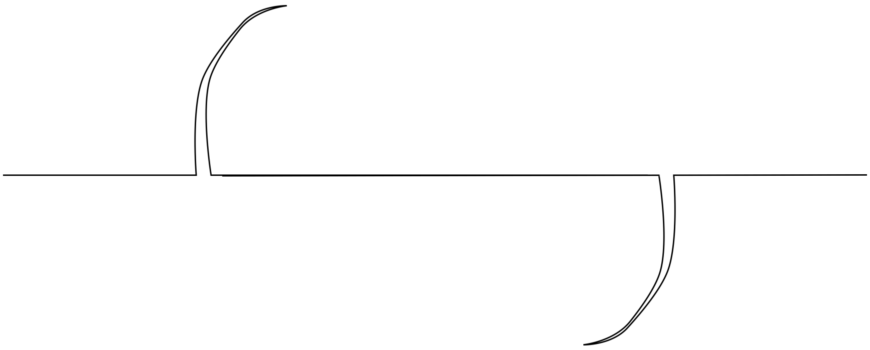

FIGURE 1. This example shows that the ϵ numbers are strictly smaller than the β numbers.

One should compare Theorem 4.2 to a theorem of Bishop-Jones [**BJ94**], who obtained an analogue of the ϵ^2-conjecture for the β-numbers introduced by Jones [**J90**]. Define

$$\beta_{\infty,\Gamma}(B(x,r)) = \frac{1}{r} \inf_{\substack{L \text{ a line} \\ L\cap B(x,r)\neq\varnothing}} \sup_{y\in\Gamma\cap B(x,r)} \operatorname{dist}(y,L). \tag{4.2}$$

This is just the 'ball' version of the quantity given for dyadic cubes in (1.10) of Chapter 1.

It is immediate from the definitions that

$$\epsilon(x,r') \lesssim \beta_{\infty,\Gamma}(B(x,r)) \text{ for } r/2 < r' < r \tag{4.3}$$

whenever x is a tangent point of Γ and we choose r sufficiently small. Bishop and Jones (Theorem 2 in [**BJ94**]) proved that, up to a set of $\mathcal{H}^1$-measure zero, Γ has a tangent at $x \in \Gamma$ if and only if

$$\int_0^1 \beta_{\infty,\Gamma}(B(x,r))^2 \frac{dr}{r} < +\infty. \tag{4.4}$$

In view of (4.3), this result gives another proof of the 'if' direction of Conjecture 4.1, but is strictly weaker than Theorem 4.2 in the opposite direction. See Figure 1. Of course, as a consequence of Theorem 4.2 and Theorem 2 of [**BJ94**], we have that, up to a set of $\mathcal{H}^1$-measure zero, $\mathcal{E}(x) < \infty$ if and only if (4.4) holds at $x \in \Gamma$.

1.1. Outline. When Conjecture 4.1 is discussed on p. 141 of [**DS91**], it is described how the coefficients $\epsilon(x,r)$ are not sufficiently stable to apply the methods developed in [**DS91**], even if one only wishes to show the rectiability of an Ahlfors regular subset of $\{x \in \Gamma : \mathcal{E}(x) < \infty\}$.

An important first step in the proof below, also bearing in mind the remark above, is the introduction of an auxiliary square function, denoted by α, which is smoother and more stable. There are three fundamental properties of α. First, it is controlled by ε, and thus it is finite whenever ε is. Second, it has an analytic kernel, and third it gives information on the geometry on the *whole* ball, rather than just on its boundary (as is the case for ε). In particular, the second property can be used to show, roughly speaking, that the set of points where $\alpha \equiv 0$ is an analytic arc S. With a lot of technical work and not forgetting about the finiteness of ε (thus the original square function will be used *directly* in the arguments below), one can then show that S is, in fact, a line. Through a limiting argument, we can conclude that the set of points where α *and* ε are finite get flatter and flatter as we zoom in through the scales. This gives the geometric control expressed in Main Lemma 4.5.

The conclusions of Main Lemma 4.5, together with some Fourier estimates which give a very good control on α on Lipschitz graphs, are then used to prove Theorem 4.2 through a well known scheme of David and Semmes ([**DS91**]) and Legèr ([**Lé**]); this is expressed in Main Lemma 4.7.

Acknowlegement. The work leading to [**JTV**] began at the workshop *Harmonic Analysis in Nonhomogeneous Settings and Applications* at the University of Birmingham in June 2019. I would like to thank María Carmen Reguera for organising this workshop and for finding a room

at the very last minute. I, together with B. Jaye and X. Tolsa benefited from some discussions with Jonas Azzam in the early stage of the project.

2. Preliminaries

2.1. Cubes and balls. In this chapter, we will use the system of 'dyadic cubes' for general sets as given in Theorem 5.3, see the Appendix.

Note that in this chapter, by balls we really mean discs in the plane. Given a ball B, we will denote its radius by

$$r(B).$$

2.2. Jordan domains. A domain is a connected open set. We call a domain Ω^+ a Jordan domain if its boundary $\Gamma = \partial\Omega^+$ is a Jordan curve. In this case (by the Jordan curve theorem), $\Omega^- = \mathbb{R}^2 \backslash \Omega^+$ is also a Jordan domain.

2.3. Measures. Throughout the chapter, by a measure we shall mean a non-negative locally finite Borel measure.

For $C_0 > 0$, a measure μ has C_0-linear growth if

$$\mu(B(x,r)) \leq C_0 r \text{ for all } x \in \mathbb{R}^2 \text{ and } r > 0.$$

For a ball $B \subset \mathbb{R}^2$, we write

$$\Theta_\mu(B) = \frac{\mu(B)}{r(B)}.$$

This should be understood as a kind of 1-dimensional density of μ over B.

2.4. 2-sided corkscrew open sets. Let $\Omega \subset \mathbb{R}^2$ be an open set. We say that Ω satisfies the c-corkscrew condition (or just, corkscrew condition) if there exists some $c > 0$ such that for all $x \in \partial\Omega$ and all $0 < r < \operatorname{diam}(\Omega)$ there exists some ball $B \subset \Omega \cap \overline{B(x,r)}$ with $r(B) \geq cr$.

We say that Ω satisfies the 2-sided (c-)corkscrew condition if both Ω and $\mathbb{R}^2 \setminus \overline{\Omega}$ satisfy the (c-)corkscrew condition.

We say that $\Omega \subset \mathbb{R}^2$ is a 2-sided corkscrew open set (or domain) if it is an open set (or domain) that satisfies the 2-sided corkscrew condition.

For example, quasicircles are 2-sided corkscrew domains. Indeed, quasicircles are simply connected 2-sided corkscrew domains that satisfy a Harnack chain condition, according to Peter Jones (see Theorem 2.7 in [JK]), or in other words, they are the same as planar simply connected NTA domains. On the other hand, it is easy to check that there are simply connected 2-sided corkscrew domains which are not quaiscircles.

2.5. Cones and tangents. For a point$x \in \mathbb{R}^2$, a unit vector u, and an aperture parameter $a \in (0,1)$ we consider the two sided cone with axis in the direction of u defined by

$$X_a(x,u) = \big\{y \in \mathbb{R}^2 : |(y-x)\cdot u| > a|y-x|\big\}.$$

Given an open set $\Omega^+ \subset \mathbb{R}^2$ and $x \in \partial\Omega^+$, we say that $\partial\Omega^+$ has a *tangent* at x, and that x is a *tangent point* for $\partial\Omega^+$ if there exists a unit vector u such that, for all $a \in (0,1)$, there exists some $r > 0$ such that

$$\partial\Omega^+ \cap X_a(x,u) \cap B(x,r) = \emptyset$$

and moreover, one component of $X_a(x,u) \cap B(x,r)$ is contained in Ω^+ and the other in $\Omega^- = \mathbb{R}^2 \setminus \overline{\Omega^+}$. The line L orthogonal to u through x is called a tangent line at x. Note that this notion of tangent is associated with the domain Ω^+, and it would be more appropriate to say that L is a tangent for Ω^+.

3. Smoother square functions

Several smoother versions of the Carleson square function play an important role in our analysis, as one can see by the statements of Main Lemmas 4.5 and 4.7 in the next section. In this section we show that these smoother square functions are controlled by the Carleson square function (with the addition of an absolute constant).

Suppose that $\Omega^+ \subset \mathbb{R}^2$ is an open set, $\Gamma = \partial\Omega^+$ and $\Omega^- = \mathbb{R}^2 \backslash \overline{\Omega^+}$.

First denote

$$\alpha^+(x,r) = \left|\frac{\pi}{2} - \frac{1}{r^2}\int_{\Omega^+} e^{-|y-x|^2/r^2}\,dy\right|,$$

and set

$$\mathcal{A}(x)^2 = \int_0^1 \alpha^+(x,r)^2\,\frac{dr}{r}.$$

More generally, for a non-negative smooth function $\varphi : \mathbb{R} \to [0,\infty)$ satisfying

$$\int_0^\infty \varphi(t)t\sqrt{\log(e+t)}dt < \infty,$$

set $\psi(x) = \varphi(|x|)$. Consider

$$\mathfrak{a}_\psi(x,r) = \left|c_\psi - \frac{1}{r^2}\int_{\Omega^+}\psi\Big(\frac{x-y}{r}\Big)dy\right|, \quad \text{where } c_\psi = \int_{\mathbb{R}^2_+}\psi\Big(\frac{y}{r}\Big)dy,$$

and define

$$\mathcal{A}^1_\psi(x)^2 = \int_0^1 \mathfrak{a}_\psi(x,r)^2\frac{dr}{r}.$$

LEMMA 4.4. *There is an absolute constant $C_{1,\psi}$ such that for every $x \in \mathbb{R}^2$, every $R > 0$, and $M \geq 1$, we have*

$$\int_0^R \mathfrak{a}_\psi(x,r)^2\frac{dr}{r} \leq C_{1,\psi}\int_0^{MR}\epsilon(x,r)^2\frac{dr}{r} + 2\int_M^\infty \varphi(t)\,t\,\Big(\log^+\frac{t}{M}\Big)^{1/2}dt.$$

PROOF. Observe that, by integrating polar coordinates centered at x,

$$\begin{aligned}\mathfrak{a}_\psi(x,r) &= \left|c_\psi - \frac{1}{r^2}\int_0^\infty \varphi\Big(\frac{s}{r}\Big)\mathcal{H}^1(\partial B(x,s)\cap\Omega^+)\,ds\right| \\ &= \frac{1}{r^2}\left|\int_0^\infty \varphi\Big(\frac{s}{r}\Big)\big(\pi s - \mathcal{H}^1(\partial B(x,s)\cap\Omega^+)\big)\,ds\right|.\end{aligned} \tag{4.5}$$

Next recall that $I_\pm(x,s)$ are the longest arcs in $\Omega^\pm \cap \partial B(x,s)$, so

$$I^+(x,s) \subset \partial B(x,s)\cap\Omega^+ \subset \partial B(x,s)\setminus I^-(x,s),$$

and consequently

$$\mathcal{H}^1(I^+(x,s)) \leq \mathcal{H}^1(\partial B(x,s)\cap\Omega^+) \leq 2\pi s - \mathcal{H}^1(I^-(x,s)).$$

Subtracting πs from this inequality easily yields that

$$\big|\pi s - \mathcal{H}^1(\partial B(x,s)\cap\Omega^+)\big| \leq s\,\epsilon(x,s),$$

which, when plugged into (4.5) yields

$$\mathfrak{a}_\psi(x,r) \leq \frac{1}{r^2}\int_0^\infty \varphi\Big(\frac{s}{r}\Big)\epsilon(x,s)\,s\,ds. \tag{4.6}$$

Squaring both sides of (4.6) and integrating over $r \in (0,R)$ yields,

$$\begin{aligned}\left(\int_0^R \mathfrak{a}_\psi(x,r)^2\frac{dr}{r}\right)^{1/2} &\leq \left(\int_0^R\left(\frac{1}{r^2}\int_0^\infty \varphi\Big(\frac{s}{r}\Big)\epsilon(x,s)\,s\,ds\right)^2\frac{dr}{r}\right)^{1/2} \\ &= \left(\int_0^R\left(\int_0^\infty \varphi(t)\epsilon(x,tr)\,t\,dt\right)^2\frac{dr}{r}\right)^{1/2} \\ &\overset{\text{Minkowski's inequality}}{\leq} \int_0^\infty\left(\int_0^R \epsilon(x,tr)^2\frac{dr}{r}\right)^{1/2}\varphi(t)\,t\,dt \\ &= \int_0^\infty\left(\int_0^{tR}\epsilon(x,u)^2\frac{du}{u}\right)^{1/2}\varphi(t)\,t\,dt.\end{aligned}$$

For $t \geq 1$ and some $M > 1$, we split

$$\int_0^{tR}\epsilon(x,u)^2\frac{du}{u} \leq \int_0^{MR}\epsilon(x,u)^2\frac{du}{u} + \int_{MR}^{tR}\frac{du}{u} = \int_0^{MR}\epsilon(x,u)^2\frac{du}{u} + \log^+\frac{t}{M}.$$

This bound certainly also holds for $t \in (0, 1)$, so we get

$$\int_0^\infty \left(\int_0^{tR} \epsilon(x, u)^2 \frac{du}{u} \right)^{1/2} \varphi(t)\, t\, dt \le \left(\int_0^{MR} \epsilon(x, u)^2 \frac{du}{u} \right)^{1/2} \int_0^\infty \varphi(t)\, t\, dt + \int_M^\infty \varphi(t)\, t \left(\log^+ \frac{t}{M} \right)^{1/2} dt,$$

and the lemma follows. □

4. The two main lemmas and the proof of the main theorem

Having introduced an array of square functions, we may now state the primary two technical results of this chapter. The chapter splits into two essentially disjoint parts, which use very different techniques.

Part I. The first part of the chapter concerns the use of compactness arguments to show, roughly speaking, that the curve Γ must be quite flat near points where the Carleson square function is finite.

Main Lemma 4.5. *Let $\Omega^+ \subset \mathbb{R}^2$ be either a Jordan domain or a two sided corkscrew open set, let $\Gamma = \partial\Omega^+$, and let μ be a measure with 1-linear growth supported on Γ. Let B be a ball centered at Γ such that*

$$\mu(B) \ge \theta r(B),$$

for some $\theta \in (0, 1)$. Given any $\epsilon > 0$, there exists $\delta \in (0, 1)$, depending on θ and ϵ (and the two sided corkscrew parameter in that case), such that if

$$\int_{7B} \int_0^{7r(B)} [\epsilon(x, r)^2 + \alpha^+(x, r)^2]\, d\mu(x) \frac{dr}{r} \le \delta\, \mu(7B),$$

then

$$\beta_{\infty,\Gamma}(B) \le \epsilon.$$

Remark 4.6. Note that Main Lemma 4.5 gives a partial control on the flatness not only of $\mathrm{spt}(\mu)$, but of Γ itself. This will be fundamental in the second part of the chapter, specifically in the construction of the approximating Lipschitz graph (see also Main Lemma 4.7 below).

The proof of Main Lemma 4.5 is considerably easier in the case of 2-sided corkscrew open sets, since these sets are rather stable under natural limit operations (see Lemma 4.19). Jordan domains do not have similar stability properties and so the analysis is much more delicate. However, the case of the 2-sided corkscrew open set is nevertheless very instructive, as a key part of our analysis is that, if Ω^+ is a Jordan domain, then at points and scales where μ has a lot of mass, and the Carleson square function is small, one can find corkscrew balls (Lemma 4.23). This property, which is much weaker than the two sided corkscrew condition insofar as it tells us nothing about Γ at points where μ has little mass, is still sufficient for us to prove Main Lemma 4.5 with a considerable amount of additional work.

Part II. The second part of this chapter is concerned with improving the local flatness which is provided by Main Lemma 4.5 into a rectifiability property. For this we work with the general scheme introduced by David and Semmes [**DS91**] and extended to the non-homogeneous context by Legér [**Lé**]. In fact we will not require the full strength of the Carleson square function, but rather a smoother square function.

Main Lemma 4.7. *Let $\Omega^+ \subset \mathbb{R}^2$ be an open set, and let $\Gamma = \partial\Omega^+$. Fix $c_0 \in (0, 1)$, $\theta > 0$ and $\epsilon > 0$. Let B_0 be a ball centered at Γ and let μ be a measure with 1-linear growth supported on $\Gamma \cap \overline{B_0}$ satisfying the following conditions:*

- *$\mu(B_0) \ge c_0 r(B_0)$.*
- *$\beta_{\infty,\Gamma}(B) \le \epsilon$ for any ball B centered at Γ such that $\mu(B) \ge \theta\, r(B)$.*
- *For a radial function $\psi \in C^\infty(\mathbb{R}^2)$ with $\mathbb{1}_{B(0,1)} \le \psi \le \mathbb{1}_{B(0,1.1)}$, it holds that*

$$\int_{\mathbb{R}^d} \int_0^{r(B_0)/\epsilon} \mathfrak{a}_\psi(x, r)^2 \frac{dr}{r}\, d\mu(x) \le \epsilon \text{ for every } x \in \mathrm{spt}(\mu).$$

If θ is small enough in terms of c_0, and ϵ is small enough in terms of θ and c_0, then there exists a Lipschitz graph Λ with slope at most $1/10$ such that

$$\mu(\Lambda) \geq \frac{1}{2}\mu(B_0).$$

The key property of the square function generated by the coefficients $\mathfrak{a}_\psi$ which enables a Legér type construction are the Fourier estimates carried out in Section 9, see in particular Lemma 4.34. Subsequently, we carry out the construction itself, which has several subtleties due to the nature of our particular square function.

4.1. The proof of Theorem 4.2. Before beginning the proof we recall some basic facts about densities: For a set $E \subset \mathbb{R}^2$ we set

$$\Theta^{1,*}(x, E) = \limsup_{r\to 0} \frac{\mathcal{H}^1(E \cap B(x, r))}{2r}, \quad \Theta^1_*(x, E) = \liminf_{r\to 0} \frac{\mathcal{H}^1(E \cap B(x, r))}{2r}.$$

For the proof of the following simple lemma, see for example [**Mat95**, Theorem 6.1].

LEMMA 4.8. *If $E \subset \mathbb{R}^2$ satisfies $\mathcal{H}^1(E) \in (0, \infty)$, then*

$$\frac{1}{2} \leq \Theta^{1,*}(x, E) \leq 1 \text{ for } \mathcal{H}^1\text{-a.e. } x \in E.$$

We will also require two simple properties of Lipschitz graphs. Suppose that Λ is a Lipschitz graph in $\mathbb{R}^2$, and $F \subset \Lambda$, then for $\mathcal{H}^1$-a.e. $x \in F$

- F has a tangent at x, and
- $\Theta^1_*(x, F) = \Theta^{1,*}(x, F) = 1$.

Both properties follow easily from Lebesgue's theorem on the almost everywhere differentiability of absolutely continuous functions on the real line.

Proof of Theorem 4.2 using the Main Lemmas 4.5 and 4.7. We have to show that the set $G = \{x \in \Gamma : \mathcal{E}(x) < \infty\}$ is rectifiable and that for $\mathcal{H}^1$-a.e. $x \in G$ there exists a tangent to Γ.

Standard arguments, see, for instance [**Mat95**, Lemma 15.13], yield the rectifiability of the set G from the existence of tangents to $\Gamma \supset G$ at $\mathcal{H}^1$-almost every point of G. Therefore our goal is to prove the statement about the existence of tangents.

For the sake of contradiction, suppose that the subset $F_0 \subset G$ of those points $x \in G$ which are not tangent points for Γ has positive $\mathcal{H}^1$ measure. Consider a subset $F \subset F_0$ such that $0 < \mathcal{H}^1(F) < \infty$.

Since the Carleson square function $\mathcal{E}(x)^2 < \infty$ for $\mathcal{H}^1$-a.e. $x \in F$, we have from Lemma 4.4 that

$$\int_0^1 [\epsilon(x, r)^2 + \alpha^+(x, r)^2 + \mathfrak{a}_\psi(x, r)^2]\frac{dr}{r} < \infty \text{ for } \mathcal{H}^1\text{-a.e. } x \in F,$$

where ψ is the function from Main Lemma 4.7.

By replacing F by a subset with positive $\mathcal{H}^1$ measure if necessary, we may assume that

$$\lim_{s\to 0} \int_0^s \left(\epsilon(x, r)^2 + \alpha^+(x, r)^2 + \mathfrak{a}_\psi(x, r)^2\right) \frac{dr}{r} = 0 \quad \text{uniformly in } F, \tag{4.7}$$

and

$$\mathcal{H}^1(B(x, r) \cap F) \leq 3r \quad \text{for all } x \in F,\ r > 0.$$

(This second inequality is a consequence of the fact that $\Theta^{1,*}(x, F) \leq 1$ for $\mathcal{H}^1$-a.e. $x \in F$.)

For the choice $c_0 = 1/9$, pick $\theta > 0$ and then $\epsilon \in (0, \theta)$ small enough positive numbers so that Main Lemma 4.7 is applicable. Then choose $\delta > 0$ small enough so that Main Lemma 4.5 is applicable with the choice c_0 replaced by θ.

Let R be small enough so that

$$\int_0^{7R/\epsilon} \left(\epsilon(x, r)^2 + \alpha^+(x, r)^2 + \mathfrak{a}_\psi(x, r)^2\right) \frac{dr}{r} \leq \min(\delta, \epsilon). \tag{4.8}$$

for all $x \in F$.

Denote $\mu = \frac{1}{3}\mathcal{H}^1|_F$. Then μ has 1-linear growth. Recalling that $\Theta^{1,*}(x, F) \geq 1/2$ for $\mathcal{H}^1$-a.e. $x \in F$, we can find a ball B_0 centered at F with radius smaller than R such that $\mu(B_0) \geq r(B_0)/9 = c_0 r(B_0)$.

We look to apply Main Lemma 4.7 with the measure $\nu = \mu|_{B_0}$ (which satisfies $\nu(B_0) \geq c_0 r(B_0)$). Notice that if B is a ball with $\nu(B) \geq \theta r(B)$, then certainly $B \cap B_0 \neq \varnothing$, and $r(B) \leq r(B_0)/\theta \leq r(B_0)/\epsilon$. Consequently, from (4.8) we infer that

$$\int_{7B} \int_0^{7r(B)} [\epsilon(x,r)^2 + \alpha^+(x,r)^2]\, d\mu(x) \frac{dr}{r} \leq \delta\, \mu(7B),$$

but trivially have $\mu(B) \geq \theta r(B)$, and so Main Lemma 4.5 yields that $\beta_{\infty,\Gamma}(B) \leq \epsilon$.

On the other hand, it is also immediate from (4.8) that

$$\int_{\mathbb{R}^d} \int_0^{r(B_0)/\epsilon} \mathfrak{a}_\psi(x,r)^2 \frac{dr}{r} d\nu(x) \leq \epsilon \nu(\mathbb{R}^d).$$

Consequently, we may apply Main Lemma 4.7 to find a Lipschitz graph Λ such that the set $F_1 = F \cap \Lambda$ satisfies $\mathcal{H}^1(F_1) > 0$.

As a consequence, for $\mathcal{H}^1$-a.e. $x \in F_1$, we have

$$\Theta^1_*(x, F_1) = 1 \text{ and } F_1 \text{ has a tangent at } x. \tag{4.9}$$

We claim that, every $x \in F_1$ satisfying (4.9) the tangent line for F_1 at x is also a tangent to Γ.

To verify the claim, we will appeal to Main Lemma 4.5. Fix $x \in F_1$ satisfying (4.9). Observe that (4.7), along with the condition $\Theta^1_*(x, F_1) = 1$, ensure that for any $\epsilon > 0$ we can find $r_0 > 0$ such that for every $r < r_0$ we can apply Main Lemma 4.5 with the measure μ and the ball $B_0 = B(x,r)$ (with the constant c_0 equal to, say, $1/4$). Therefore,

$$\lim_{r\to 0} \beta_{\infty,\Gamma}(B(x,r)) = 0. \tag{4.10}$$

Now, let u be a unit vector orthogonal to the tangent line at x to F_1. Observe that, for every $a \in (0,1)$, $\Gamma \cap X_a(x,u) \cap A(x, r/2, r) = \varnothing$ for all sufficiently small $r > 0$, since otherwise $\beta_{\infty,\Gamma}(B(x,r)) \geq c(a) > 0$ for all r small enough, contradicting (4.10).

But the condition $\Gamma \cap X_a(x,u) \cap A(x,r/2,r) = \varnothing$ for all r small enough clearly implies that $\Gamma \cap X_a(x,u) \cap B(x,r) = \varnothing$ for all r small enough, and so Γ has a tangent at x. Therefore our claim follows, and this in turn clearly contradicts the fact that the points in F_0, and thus the ones in F_1, are not tangent points for Γ. $\square$

Part I: Flatness via compactness arguments

5. Sequences of measures, boundaries and domains

5.1. Weak convergence of measures. We say that a sequence of (Radon) measures μ_j converges weakly to a measure μ if

$$\int_{\mathbb{R}^d} f\, d\mu_j = \int_{\mathbb{R}^d} f\, d\mu \text{ for every } f \in C_0(\mathbb{R}^2),$$

where $C_0(\mathbb{R}^2)$ denotes the continuous functions with compact support. The following compactness theorem for Radon measures can be deduced from the Riesz representation theorem (see Theorem 2.5.13 in [**Fe69**]) and a diagonal argument. See [**Mat95**], Theorem 1.23, or [**Si14**], Theorem 4.16.

Lemma 4.9. *If μ_j is a sequence of measures in $\mathbb{R}^2$ satisfying $\sup_j \mu_j(B(0,R)) < \infty$ for any $R \in (0,\infty)$, then μ_j has a weakly convergent subsequence.*

It is easy to see that weak limits are lower-semicontinuous on open sets and upper-semicontinuous on compact sets.

Bringing our observations together, we arrive at the following result.

Lemma 4.10. *Fix $C_0, c_0 \in (0,\infty)$. Fix a ball $B_0 \subset \mathbb{R}^2$. Suppose that μ_j is a sequence of measures with C_0-linear growth such that $\mu_j(\overline{B_0}) \geq c_0 r(B_0)$ for every j. Then there is a subsequence μ_{j_k} of the measures which converges weakly to a measure μ with C_0-linear growth satisfying $\mu(\overline{B_0}) \geq c_0 r(B_0)$.*

This lemma is an immediate consequence of Theorem 1.24 in [**Mat95**].

We next establishing some basic facts about convergence of sets.

5.2. Convergence of sets. For $B \subset \mathbb{R}^2$ and $x \in \mathbb{R}^2$, set

$$\operatorname{dist}(x, B) = d(x, B) = \inf_{b \in B} |x - b|.$$

For non-empty sets A and B, we define the excess of A over B to be the quantity

$$\operatorname{excess}(A, B) = \sup_{x \in A} d(x, B),$$

and put $\operatorname{excess}(\varnothing, B) = 0$ while $\operatorname{excess}(A, \varnothing)$ is left undefined.

Observe that $\operatorname{excess}(A, B) < \epsilon$ means that the open ϵ-neighbourhood of B contains A.

The Hausdorff distance between A and B is given by

$$\operatorname{dist}_H(A, B) = \max\{\operatorname{excess}(A, B), \operatorname{excess}(B, A)\}.$$

For compactness arguments we will require a notion of local convergence. To this end, we follow [**BL**] and introduce the relative Walkup-Wets distance. For non-empty sets A, B, we define, for $R > 1$,

$$d_R(A, B) = \max\big\{\operatorname{excess}(A \cap \overline{B(0, R)}, B), \operatorname{excess}(B \cap \overline{B(0, R)}, A)\big\}.$$

(The reader should not be concerned that the quantity d_R need not satisfy the triangle inequality.)

Observe that

$$d_{R_1}(A, B) \leq d_{R_2}(A, B) \leq \operatorname{dist}_H(A, B) \text{ if } R_2 \geq R_1. \tag{4.11}$$

Definition 4.11 (Local Convergence). A sequence of non-empty sets E_j converge locally to a non-empty set E (written $E_j \to E$ locally) if, for any $R > 0$,

$$\lim_{j \to \infty} d_R(E_j, E) = 0.$$

We refer the reader to Section 2 of [**BL**] for a more thorough introduction to this notion of convergence. In variational analysis, this notion of convergence is called convergence in the Attouch-Wets topology.

Lemma 4.12. *If E_j are non-empty closed sets that converge locally to a non-empty closed set E, then*

(1) a compact set K satisfies $K \cap E = \varnothing$ if and only if there is a neighbourhood of K that has empty intersection with E_j for all sufficiently large j, and
(2) if the sets E_j are contained in a fixed compact set, then E_j converge locally to E if and only if E_j converges to E in the Hausdorff distance.

Proof. Both properties are straightforward consequences of the local convergence, so we shall only verify the 'if' direction of (1). If $K \cap E = \varnothing$, there exists $r > 0$ such that K_δ, the δ-neighbourhood of K, satisfies

$$\inf_{x \in K_\delta,\, y \in E} |x - y| > \delta.$$

But then there exists $R > 0$ such that $K_{2\delta} \subset B(0, R)$. But then

$$d_{2R}(E_j, E) < \delta \text{ for sufficiently large } j,$$

so the open δ-neighbourhood of E contains $E_j \cap B(0, 2R)$ for sufficiently large j. Consequently, $K_\delta \cap E_j = K_\delta \cap E_j \cap B(0, 2R) = \varnothing$ for sufficiently large j. □

We next state a basic compactness result.

Lemma 4.13. *Suppose that E_j is a sequence of closed sets in $\mathbb{R}^2$ that intersect $\overline{B(0, 1)}$. Then there is a subsequence E_{j_k} that converges locally to a closed set $E \subset \mathbb{R}^2$ (satisfying $E \cap \overline{B(0, 1)} \neq \varnothing$).*

This statement can be proved by modifying the usual proof of the relative compactness of a sequence of closed subsets of a compact metric in the Hausdorff topology, see also Theorem 2.5 of [**BL**] and references therein.

Let us now fix open sets $\Omega_j^+ \subset \mathbb{R}^d$ with boundary $\Gamma_j = \partial\Omega_j^+$. We write $\Omega_j^- = \mathbb{R}^2 \backslash \overline{\Omega_j^+}$. Throughout this chapter, we will always be working in situations where also

$$\Gamma_j = \partial\Omega_j^-. \tag{4.12}$$

LEMMA 4.14. *Let $\{\Omega_j^+\}_j$ be a sequence of open sets in the plane. Set $\Omega_j^- = \mathbb{R}^2 \setminus \overline{\Omega_j^+}$ and suppose that $\Gamma_j = \partial\Omega_j^+$ satisfies (4.12). Suppose there are closed sets G^+, G^-, G_0 satisfying*

$$\overline{\Omega_j^\pm} \to G^\pm \quad \textit{and} \quad \Gamma_j \to G_0 \quad \textit{locally.}$$

Then

(1) *The limit sets G^+, G^-, G_0 satisfy*
$$G^+ \cup G^- = \mathbb{R}^2, \qquad G^+ \cap G^- = G_0.$$
In particular, $G^+ \setminus G_0$ and $G^- \setminus G_0$ are open.

(2) *There are functions $g^+, g^- \in L^\infty(\mathbb{R}^2)$ such that, for a subsequence $\Omega_{j_k}^\pm$*
$$\mathbb{1}_{\Omega_{j_k}^\pm} \to g^\pm \quad \textit{weakly} * \textit{ in } L^\infty(\mathbb{R}^2),$$
where
$$g^+ = 1 \textit{ in } G^+ \setminus G_0 \qquad \textit{and} \qquad g^+ = 0 \textit{ in } G^- \setminus G_0.$$

PROOF. Suppose that $y \in \mathbb{R}^2 \setminus G^+ \cup G^-$. Then, since G^+ and G^- are closed, there exists an $r > 0$ with $B(y,r) \subset \mathbb{R}^2 \setminus G^+ \cup G^-$. In particular, there exists a compact set K which is disjoint from $G^+ \cup G^-$. But note that for all j, we have that $\overline{\Omega_j^\pm} \cap \supset K$, since $\overline{\Omega_j^+} \cup \overline{\Omega_j^-} = \mathbb{R}^2$ by definition. This contradicts conclusion (1) of Lemma 4.12. This proves the first assertion of (1) in this lemma. As for the second assertion, since $\Gamma_j = \partial\Omega_j^+ = \partial\Omega_j^- \subset \overline{\Omega_j^+} \cap \overline{\Omega_j^-}$, the from the same line or argument as above, we have $G_0 \subset G^+ \cap G^-$. On the other hand, if $x \in G^+ \cap G^-$, then for any $\epsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that for $j \geq j_0$,

$$\operatorname{dist}\left(x, \overline{\Omega_j^+}\right) \leq \epsilon \quad \text{and} \quad \operatorname{dist}\left(x, \overline{\Omega_j^-}\right) \leq \epsilon.$$

That is, there exist $y_j^\pm \in \Omega_j^\pm$ such that $|x - y_j^\pm| \leq \epsilon$. There exists some $z \in \Gamma_j$ in the segment $[y_j^+, y_j^-]$, and thus $|x - z| \leq \epsilon$ and $\operatorname{dist}(x, \Gamma_j) \leq \epsilon$. Since this holds for all j big enough, we deduce that x belongs to the limit in the Attouch-Wets topology of $\{\Gamma_j\}_k$, that is, to G_0.

To see the openness of $G^+ \setminus G_0$, note that $\mathbb{R}^2 = (G^+ \setminus G_0) \cup G^-$ is a disjoint union. Thus $G^+ \setminus G_0 = \mathbb{R}^2 \setminus G^-$ is open. Analogously, $G^- \setminus G_0 = \mathbb{R}^2 \setminus G^+$ is open.

We now turn our attention to verifying (2). The existence of $g^\pm \in L^\infty(\mathbb{R}^2)$ such that, for a subsequence Ω_{j_k}, $\mathbb{1}_{\Omega_{j_k}^\pm} \to g^\pm$ weakly $*$ in $L^\infty(\mathbb{R}^2)$ is a standard consequence of the Banach-Alouglu theorem. Now consider a continuous function φ compactly supported on $G^+ \setminus G_0$. Recall that $G^+ \setminus G_0$ is open and $G^+ \setminus G_0 = \mathbb{R}^2 \setminus G^-$. Consequently, property (1) of Lemma 4.12 ensures that there exists some $\epsilon > 0$ such that, for all k big enough,

$$\operatorname{dist}\left(\operatorname{supp}\varphi, \overline{\Omega_{j_k}^-}\right) \geq \epsilon.$$

In particular, $\operatorname{supp}\varphi \subset \Omega_{j_k}^+$ for all k big enough, which implies that

$$\int_{\mathbb{R}^2} \mathbb{1}_{\Omega_{j_k}^+} \varphi \, dx = \int_{\mathbb{R}^2} \varphi \, dx \qquad \text{for all } k \text{ big enough,}$$

and proves that g^+, the weak $*$ limit of $\mathbb{1}_{\Omega_{j_k}^+}$, equals 1 in $G^+ \setminus G_0$. The proof that $g^+ = 0$ in $G^- \setminus G_0$ is completely analogous. $\square$

6. Square functions and limits of domains

Throughout this section, fix a sequence of sets Ω_j^+ with $\Omega_j^- = \mathbb{R}^2 \setminus \overline{\Omega_j^+}$, such that $\Gamma_j = \partial\Omega_j^+ = \partial\Omega_j^-$ (i.e. (4.12) holds). Assume that $\overline{\Omega_j^\pm} \to G^\pm$ and $\Gamma_j \to G_0$ locally as $j \to \infty$. Consequently, the sets G^+, G^- and G_0 will satisfy the properties of Lemma 4.14.

6.1. Complementary semicircumferences and admissible pairs. It will be convenient to introduce the following definitions, which will be central to our analysis.

DEFINITION 4.15 (Complementary semicircumferences). We say that two closed semicircumferences are *complementary* if they are contained in the same circumference and their intersection consists just of their end-points.

DEFINITION 4.16 (Admissible pairs). We say that a pair of two complementary closed semicircumferences (S_1, S_2) is admissible (for the sequence of sets $\{\Omega_j^+\}_j$) if there exists a subsequence of circular arcs $I^{\pm}_{j_k} \subset \partial B(x_{j_k}, r_{j_k}) \cap \Omega^{\pm}_{j_k}$, with $x_{j_k} \in \Gamma_{j_k}$, such that $I^{+}_{j_k}$, $I^{-}_{j_k}$ converge to S_1, S_2 in Hausdorff distance, respectively.

It is immediate to check that, if (S_1, S_2) is an admissible pair, then $S_1 \subset G^+$, $S_2 \subset G^-$ and that the common center of S_1 and S_2 belongs to G_0. Consequently, we will also say that S_1 and S_2 are admissible for G^+ and G^-, respectively. We call the common center and radius of S_1, S_2 the center and radius of the pair, respectively.

Observe that, for a given $x \in G_0$, $r > 0$, there may exist more than one admissible pair of semicircumferences centered at x with radius r.

Especially when dealing with Jordan domains, we will use the fact that *the set of admissible pairs is closed in the topology of Hausdorff distance*, that is, if $\{(S_{1,i}, S_{2,i})\}$ is a sequence of admissible pairs (possibly with different centers and radii) such that $S_{1,i}$, $S_{2,i}$ converge respectively to S_1, S_2, then (S_1, S_2) is an admissible pair.

To check this fact, just take for each i a pair of arcs I_i^+, I_i^- contained in $\partial B(x_{j_i}, r_{j_i}) \cap \Omega^{\pm}_{j_i}$, with $x_i \in \partial\Omega^{\pm}_{j_i}$, for a suitable j_i such that

$$\mathrm{dist}_H(I_i^{\pm}, S_{1,i}) \leq \frac{1}{i}.$$

It is clear that the arcs I_i^+, I_i^- converge respectively to S_1, S_2 in Hasdorff distance, and thus (S_1, S_2) is an admissible pair.

6.2. A general convergence result. We set $\epsilon_j(x,r)$ and $\alpha_j^+(x,r)$ to be the coefficients $\epsilon(x,r)$ and $\alpha^+(x,r)$ associated with Ω_j^+.

Lemma 4.17. *Fix $C_0, c_0 > 0$. Let $\{\mu_j\}_j$ be a sequence of measures with C_0-linear growth supported on Γ_j converging weakly to a measure μ_0 (so μ_0 is supported in G_0, and has C_0-linear growth). Suppose B_0 is a ball with $\mu_0(\overline{B_0}) \geq c_0 r$. Further assume that, for each j, both*

$$\int_{7B_0} \int_0^{7r(B_0)} \alpha_j^+(x,r)^2 \, \frac{dr}{r} \, d\mu_j(x) \leq \frac{1}{j}\, \mu_j(7B_0), \tag{4.13}$$

and

$$\int_{7B_0} \int_0^{7r(B_0)} \epsilon_j(x,r)^2 \, \frac{dr}{r} \, d\mu_j(x) \leq \frac{1}{j}\, \mu_j(7B_0). \tag{4.14}$$

Then

(1) *There is an analytic variety Z such that* $\mathrm{supp}(\mu_0) \cap 7B_0 \subset Z \subset G_0$.
(2) *For all $x \in 7B_0 \cap \mathrm{supp}\,\mu_0$ and all $r \in (0, 7r(B_0))$ there is a pair of admissible semicircumferences which are contained in $\partial B(x,r)$.*

Proof. We may assume by scaling that $B_0 = B(0,1)$. The property (4.13) is responsible for the first conclusion, while (4.14) is responsible for the second conclusion.

Proof of (1). Recall from Lemma 4.14 that there is a subsequence of the open sets $\Omega^{\pm}_{j_k}$ whose characteristic functions converge weak-$*$ in L^∞ to functions $g^{\pm}$ with $g^+ \equiv 1$ on $G^+ \setminus G_0$ and $g^+ \equiv 0$ on $G^- \setminus G_0$.

Claim 5. *One has $\alpha_0^+(x,r) = 0$ for all $x \in 7B_0 \cap \mathrm{supp}\,\mu_0$ and all $r \in (0, 7r(B_0))$, where*

$$\alpha_0^+(x,r) = \left| \frac{\pi}{2} - \frac{1}{r^2} \int g^+(y)\, e^{-|y-x|^2/r^2}\, dy \right|. \tag{4.15}$$

Proof of Claim 5. For any $r > 0$, the mapping $x \mapsto \alpha_0^+(x,r)$ is continuous on $\mathbb{R}^2$, the claim will follow once we show that

$$\int_{7B_0} \int_0^7 \alpha_0^+(x,r)^2 r^3 \, dr \, d\mu_0(x) = 0, \tag{4.16}$$

Note that, since $r \leq 1$, (4.13) implies that

$$\int_{7B_0} \int_0^7 \alpha_{j_k}^+(x,r)^2 r^3 \, dr \, d\mu_k(x) \leq \frac{\mu_{j_k}(7B_0)}{j_k} \leq \frac{C}{k}. \tag{4.17}$$

Consider arbitrary non-negative smooth functions $\tilde{\mathbb{1}}_{7B_0}(x)$, $\tilde{\mathbb{1}}_{(0,7)}(r)$ compactly supported in $7B_0$ and $(0,7)$, respectively. Define

$$f_k(x,r) := \tilde{\mathbb{1}}_{7B_0}(x)\tilde{\mathbb{1}}_{[0,7]}(r)\, r^3 \left(\frac{1}{r^2} e^{-\left|\frac{\cdot}{r}\right|^2} * \mathbb{1}_{\Omega^+_{j_k}} - \frac{\pi}{2}\right)^2 .$$

Since $\mathbb{1}_{\Omega^+_{j_k}}$ converges weakly $*$ in $L^\infty(\mathbb{R}^2)$ to g^+, then we have that

$$f_k(x,r) \to f(x,r) \text{ pointwise},$$

where

$$f(x,r) = \tilde{\mathbb{1}}_{7B_0}(x)\tilde{\mathbb{1}}_{(0,7)}(r)\, r^3 \left(\frac{1}{r^2} e^{-\left|\frac{\cdot}{r}\right|^2} * g^+ - \frac{\pi}{2}\right)^2 .$$

Clearly, f_k is a uniformly bounded sequence on $\overline{7B_0 \times [0,7]}$ with uniformly bounded derivative[1] (it is to ensure this condition that we introduce the factor r^3 in (4.17)). Thus by the Arzelà-Ascoli Theorem, we deduce that f_k converges uniformly on compacts subsets to f, up to a subsequence which we relabel.

To prove (4.16) we write

$$\iint f\, dr\, d\mu_0(x) = \iint f\, dr\, d(\mu_0 - \mu_{j_k}) + \iint (f - f_k)\, dr\, d\mu_{j_k} + \iint f_k\, dr\, d\mu_{j_k}$$

The first integral tends to 0 as $k \to \infty$, since clearly $dr\, d\mu_{j_k}$ converges weakly to $dr\, d\mu_0$. Similarly, the second integral converges to 0 as $k \to \infty$, by the uniform convergence on compact subsets of f_k to f. As for the third integral, we see that

$$\left|\iint f_k\, dr\, d\mu_{j_k}\right| \le \int_{7B_0} \int_0^7 \alpha^+_{j_k}(x,r)^2 r^3\, dr\, d\mu_{j_k}(x) \le \frac{C}{k},$$

by (4.17). This immediately gives that

$$\iint f\, dr\, d\mu_0 = 0,$$

and since $\tilde{\mathbb{1}}_{7B_0}(x)$, $\tilde{\mathbb{1}}_{(0,7)}(r)$ are arbitrary non-negative smooth functions compactly supported in $7B_0$ and $(0,7)$ respectively, (4.16) follows. □

CLAIM 6. *For any $x \in \mathbb{R}^2$, if there exists a sequence $r_k \to 0$ such that $\alpha^+_0(x, r_k) = 0$ for all k, then $x \in G_0$.*

PROOF OF CLAIM 6. Recall that $g^+ = 1$ in $G^+ \setminus G_0$ and $g^+ = 0$ in $G^- \setminus G_0$. So, if $x \in G^+ \setminus G_0$, then it is immediate to check that

$$\lim_{r\to 0} \frac{1}{r^2} \int g^+(y)\, e^{-|y-x|^2/r^2}\, dy = \int e^{-y^2}\, dy = \pi,$$

taking also into account that $G^+ \setminus G_0$ is open. Thus $\alpha^+_0(x,r)$ is bounded away from 0 for all $r > 0$ small enough.

Similarly, if $x \in G^- \setminus G_0$, then $\lim_{r\to 0} \frac{1}{r^2} \int_{\mathbb{R}^2} g^-(y) e^{-|y-x|^2/r^2} dy = 0$, so $\alpha^+_0(x,r)$ is bounded away from 0 if r is sufficiently small. □

We now complete the proof of property (1). Set

$$Z = \bigcap_{k\ge 0} \{x \in \mathbb{R}^2 : \alpha^+_0(x, 2^{-k}) = 0\},$$

where α^+_0 is defined in (4.15). By Claims 5 and 6,

$$7B_0 \cap \operatorname{supp} \mu_0 \subset Z \subset G_0.$$

To see that Z is a real analytic variety consider

$$F := \sum_{k\ge 0} 2^{-k} \alpha^+_0(\cdot, 2^{-k})^2 = \sum_{k\ge 0} 2^{-k} \left(\frac{1}{r_k^2}\, g^+ * e^{-|\cdot|^2 2^{2k}} - \frac{\pi}{2}\right)^2 .$$

Then F is a real analytic function and $Z = F^{-1}(0)$.

[1] This guarantees (uniform) equicontinuity of the family f_k, which is needed to apply Arzelá-Ascoli.

Proof of property (2). Denote

$$E_k = \Big\{x \in \Gamma_k \cap 7B_0 : \textstyle\int_0^7 \epsilon_k(x,r)^2 \,\frac{dr}{r} \le \frac{1}{\sqrt{k}}\Big\}.$$

By Chebyshev's inequality, we have

$$\mu_k(7B_0 \setminus E_k) \le \sqrt{k} \int_{7B_0} \int_0^7 \epsilon_k(x,r)^2 \,\frac{dr}{r}\, d\mu_k(x) \le \frac{\sqrt{k}}{k}\, \mu(7B_0) = \frac{1}{\sqrt{k}}\, \mu_k(7B_0).$$

Set $\tau_k = 1 - k^{-1/4}$. For each $x \in E_k$ and $0 < r < 7$, we have

$$\frac{1}{\sqrt{k}} \ge \int_0^7 \epsilon_k(x,s)^2 \,\frac{ds}{s} \ge \int_{\tau_k r}^r \epsilon_k(x,s)^2 \,\frac{ds}{s} \ge \inf_{s\in[\tau_k r, r]} \epsilon_k(x,s)^2 \log\frac{1}{\tau_k} \approx \inf_{s\in[\tau_k r, r]} \epsilon_k(x,s)^2 \,\frac{1}{k^{1/4}}.$$

Hence, for all $r \in (0,7)$ there exists some $s_r \in [\tau_k r, r]$ such that

$$\epsilon_k(x, s_r) \lesssim \frac{1}{k^{1/8}}.$$

In particular, this implies that for each $x \in E_k$ and $0 < r < 7$, there exist disjoint arcs $I_k^+(x,s_r), I_k^-(x,s_r) \subset \partial B(x,s_r)$ satisfying

$$\mathcal{H}^1(I_k^\pm(x,s_r)) \ge \big(\pi - k^{-1/8}\big)\, r \quad \text{and} \quad I_k^\pm(x,s_r) \subset \overline{\Omega_k^\pm}. \tag{4.18}$$

Let $\widetilde{E}_k \subset E_k$ be a compact set such that $\mu_k(\widetilde{E}_k) \ge \frac{k-1}{k}\,\mu_k(E_k) \ge \frac{k-1}{k}\big(1 - \frac{1}{\sqrt{k}}\big)\mu_k(7B_0)$. Taking a subsequence if necessary, we can assume that $\mu_k|_{\widetilde{E}_k}$ converges weakly $*$ to some measure σ and that $\widetilde{E}_k$ converges in the Hausdorff metric to some compact set $F \subset \mathbb{R}^2$. In fact, since $\widetilde{E}_k \subset \Gamma_k$, we have $F \subset G_0$. Further, it is easy to check that $\operatorname{supp}\sigma \subset F$, and by (4.18) it follows that for all $x \in F$ and all $r \in (0,7)$, there exists an admissible pair with radius r and center x. It just remains to notice that $\sigma|_{7B_0} = \mu_0|_{7B_0}$, since for $f \in C_0(7B_0)$, $\Big|\int_{\widetilde{E}_k} f\,d\mu_k - \int_{7B_0} f\,d\mu_k\Big| \le \frac{\|f\|_\infty \mu(7B_0)}{\sqrt{k}}$. □

7. The case when Ω^+ is a 2-sided corkscrew open set

The objective of this section is to prove Main Lemma 4.5 in the case of a 2-sided corkscrew open set.

LEMMA 4.18. *Let $\Omega^+ \subset \mathbb{R}^2$ be a 2-sided c-corkscrew open set, let $\Gamma = \partial\Omega^+$, and let μ be a measure with C_0-linear growth supported on Γ. Let B be a ball centered at Γ such that*

$$\mu(B) \ge c_0 r(B),$$

for some $0 < c_0 \le C_0$. Given any $\epsilon > 0$, there exists $\delta > 0$ (depending on C_0, c_0, c, ϵ) such that if

$$\int_{7B} \int_0^{7r(B)} \big(\epsilon(x,r)^2 + \alpha^+(x,r)^2\big)\, d\mu(x) \frac{dr}{r} \le \delta\, \mu(7B),$$

then

$$\beta_{\infty,\Gamma}(B) \le \epsilon.$$

The next lemma shows that 2-sided corkscrew open sets enjoy nice limiting properties under Hausdorff limits.

LEMMA 4.19. *Let $\{\Omega_j^+\}_j$ be a sequence of c-corkscrew planar open sets such that $0 \in \partial\Omega_j^+$ and $\inf_j \operatorname{diam}(\Omega_j^+) > 0$. Let $\Omega_j^- = \mathbb{R}^2 \setminus \overline{\Omega_j^+}$ and $\Gamma_j = \partial\Omega_j^+$. Then the following holds.*

(1) There is a subsequence j_k so that

$$\Omega_{j_k}^\pm \to \Omega_\infty^\pm \quad \textit{and} \quad \Gamma_{j_k} \to \Gamma_\infty \quad \textit{locally.}$$

(2) The limit sets $\Omega_\infty^\pm$ are 2-sided corkscrew open sets such that $\Omega_\infty^- = \mathbb{R}^2 \setminus \overline{\Omega_\infty^+}$ and $\Gamma_\infty = \partial\Omega_\infty^+$.

(3) $\Omega_\infty^\pm$ satisfy the following: for any ball B such that $\overline{B} \subset \Omega_\infty^\pm$, then a neighbourhood of $\overline{B}$ is contained in $\Omega_{j_k}^\pm$ for k sufficiently large.

PROOF. This result is essentially known. See, for example Theorem 4.1 in [**KT**]. However, we are not working under precisely the assumptions in [**KT**], so we provide a proof for the reader building upon Lemma 4.14. First, Lemma 4.13 provides us with closed sets $G^\pm$ and G_0 and a subsequence such that $\Omega^{\pm}_{j_k} \to G^{\pm}$ and $\Gamma_{j_k} \to G_0$ locally as $k \to \infty$. Taking the subsequence j_k and the sets $G^{\pm}$ and G_0 provided in that lemma, we set $\Gamma_\infty = G_0$, $\Omega^+_\infty = G^+ \setminus G^-$ and $\Omega^-_\infty = G^- \setminus G^+$.

Fix $r \in (0, \operatorname{diam}(\Omega_\infty))$. Observe that $r < \liminf_{k\to\infty} \operatorname{diam}(\Omega_{j_k})$. If $x \in \Gamma_\infty$ then there is a sequence $x_{j_k} \in \Gamma_{j_k}$ with $\lim_{k\to\infty} x_{j_k} = x$. Since Ω_{j_k} is a two-sided c-corkscrew domain, and $r < \operatorname{diam}(\Omega_{j_k})$ for sufficiently large k, then there are $x^{\pm}_{j_k} \in \Omega^{\pm}_j$ with $|x_{j_k} - x^{\pm}_{j_k}| \le r$ and $B(x^{\pm}_{j_k}, c_0 r) \subset \Omega^{\pm}_{j_k}$ for k large enough. Passing to a further subsequence if necessary, we may assume $\lim_{k\to\infty} x^{\pm}_{j_k} = x^{\pm}$. But then $B(x^{\pm}, c_0 r) \subset G^{\pm}$ (for instance, any element of either of these balls can be obtain as a the limit of a sequence belonging to the respective sequences balls $B(x^{\pm}_{j_k}, c_0 r)$), and therefore $B(x^{\pm}, c_0 r) \subset \Omega^{\pm}_\infty$. Also notice that $|x^{\pm} - x| \le r$.

On the other hand, Property (1) from Lemma 4.14, ensures that $\mathbb{R}^2 = \Omega^+_\infty \cup \Gamma_\infty \cup \Omega^-_\infty$ and the union is disjoint, and so

$$\Gamma_\infty = \partial\Omega^+_\infty = \partial\Omega^-_\infty.$$

Combining our observations yields that Ω^+_∞ is a two-sided corkscrew open set, and additionally, $\Omega^{\pm}_{j_k} \to \Omega^{\pm}_\infty$ locally as $k \to \infty$. Therefore property (1) of the lemma is proved. Now property (3) follows from property (1) from Lemma 4.12, since $\Gamma_\infty = \partial\Omega^+_\infty = \partial\Omega^-_\infty$.

□

We next analyze what we can say about the natural limit situation given by the conclusions of Lemma 4.17, taking into account that the limit set $G_0 = \Gamma_\infty$ is the boundary of a 2-sided corkscrew open set.

LEMMA 4.20. *Let $\Omega^+ \subset \mathbb{R}^2$ be a non-empty open set, and let $\Omega^- = \mathbb{R}^2 \setminus \overline{\Omega^+}$ and $\Gamma = \partial\Omega^+$. Suppose that $\partial\Omega^- = \Gamma$ too. Let μ be a measure with C_0-linear growth supported on Γ and let B be a ball centered in $\operatorname{spt}(\mu)$. Suppose that*

- *there is an analytic variety Z with $\operatorname{spt}(\mu) \cap B \subset Z \subset \Gamma$, and*
- *for each $x \in B \cap \operatorname{spt}\mu$ and all $r \in (0, 3r(B))$, there exists two complementary half-circumferences $C^+(x,r)$, $C^-(x,r)$ with radius r and center x such that*

$$C^+(x,r) \subset \overline{\Omega^+} \quad \text{and} \quad C^-(x,r) \subset \overline{\Omega^-}.$$

Then $\Gamma \cap B$ is a segment.

We remark that the last property regarding the existence complementary half-circumferences $C^+(x,r)$, $C^-(x,r)$ is a consequence of the existence of admissible pairs.

PROOF. Since μ is non-zero and has linear growth[2], we have that $\mathcal{H}^1(Z) \ge \mathcal{H}^1(\operatorname{spt}(\mu) \cap B) > 0$. Together with the fact that $Z \neq \mathbb{R}^2$ (Ω^+ is non-empty), this implies that there exists an analytic curve S such that $\mu(S \cap \frac{1}{4}B) > 0$ (which implies that $\mathcal{H}^1(S \cap \operatorname{supp}\mu \cap \frac{1}{4}B) > 0$, because of the linear growth of μ).

We claim that S is a segment. To prove this, it suffices to show that S has vanishing curvature at any point of $\operatorname{supp}\mu \cap S$. Indeed, since this set has positive length and the curvature of a real analytic arc is locally a real analytic function (with respect the arc-length parametrization from an interval), this implies that the curvature vanishes on the whole arc S, and thereby prove that S is a segment.

To show that the curvature of S vanishes at $\operatorname{supp}\mu \cap S$, we will use the following property, which we will call the

KEY PROPERTY. *Given $x \in B \cap \operatorname{supp}\mu$ and $r \in (0, 3r(B))$, let $I \subset \partial B(x,r)$ be an arc such that $\mathcal{H}^1(I) < \pi r$ whose end-points belong both to Ω^+. Then $I \subset C^+(x,r)$, and thus $I \subset \overline{\Omega^+}$. The analogous statement holds replacing Ω^+ by Ω^- and $C^+(x,r)$ by $C^-(x,r)$.*

To verify that the key property holds, note that if I is an arc as above, then its end-points x_1, x_2 do not belong to $\overline{\Omega^-}$ (because they belong to Ω^+). This implies that $x_1, x_2 \in C^+(x,r)$, and thus either I or $\partial B(x,r) \setminus I$ is contained in $C^+(x,r)$. The latter cannot hold since $\mathcal{H}^1(\partial B(x,r) \setminus I) > \pi r = \mathcal{H}^1(C^+(x,r))$, and so we have $I \subset C^+(x,r)$.

We are ready now to show that the curvature of S vanishes at every $x \in \operatorname{supp}\mu \cap S$. Without loss of generality we assume that $x = 0$, and that the tangent to S at 0 is the horizontal axis.

[2]See for example Lemma 2.8 in [To14].

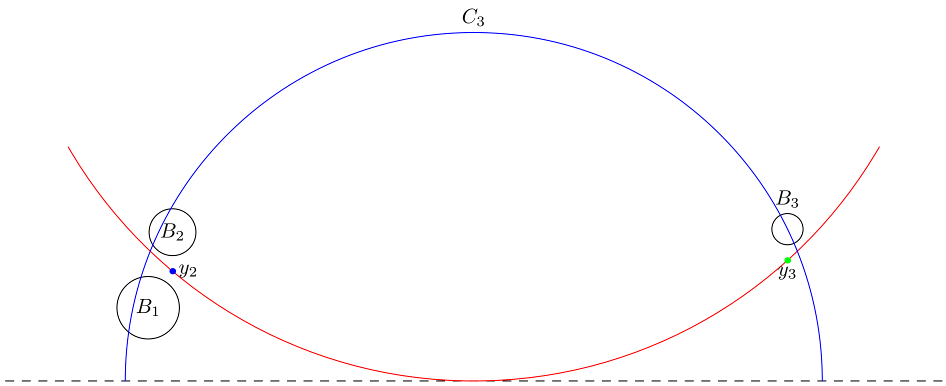

FIGURE 2. The figure depicts the first case, where B_1 is below the red curve S. (The ratio of the radii of the balls B_1, B_2, B_3 is not to scale, the reader should think of $r(B_1) \gg r(B_2) \gg r(B_3)$.)

Seeking for a contradiction, suppose that S is strictly convex at 0 (i.e., if S equals the graph of the real analytic function $g : (-\delta, \delta) \to \mathbb{R}$ in a neighborhood of 0, then $g''(0) > 0$).

Let z_1, z_2 be the two end-points of S, and let

$$d_0 = \frac{1}{2} \min_{i=1,2} \operatorname{dist}(x, z_i).$$

Let $r \in (0, d_0/2)$ be small enough so that $B(x,r) \cap S \setminus \{x\} \subset \mathbb{R}^2_+$ where $\mathbb{R}^2_+$ is the *open* upper half plane. We also assume that any point in $S \cap B(x,r)$ is at most $r/1000$ away from the the horizontal axis. Let $y_1 \in S \cap \partial B(x, r/2)$. Since $\Gamma = \partial\Omega^+$ (and Ω^+ is an open set) there exists some ball $B_1 \subset \Omega^+$ satisfying

$$\operatorname{dist}(y_1, B_1) + r(B_1) \le \frac{1}{10} \operatorname{dist}(y_1, \mathbb{R}^2_-).$$

Let C_1 be the circumference centered at x and passing through the center of B_1, and let y_2 the point belonging to $S \cap C_1$ which is closest to y_1 (if r is small enough, the set $S \cap C_1$ consist of two points by strict convexity). Using now that $\Gamma = \partial\Omega^-$, there exists some ball $B_2 \subset \Omega^-$ satisfying

$$\operatorname{dist}(y_2, B_2) + r(B_2) \le \frac{1}{10} \min\big(\operatorname{dist}(y_2, \mathbb{R}^2_-), r(B_1)\big).$$

Now let C_2 be the circumference centered at x and passing through the center of B_2, and let y_3 the point belonging to $S \cap C_2$ which is farther from y_2 (if r is small enough, the set $S \cap C_2$ consists of two points).

We distinguish now two cases. In the first one we suppose that B_2 is above B_1 (this happens if B_1 is below S), see Figure 2. Then, using again that $\Gamma = \partial\Omega^+$, there exists some ball $B_3 \subset \Omega^+$ satisfying

$$\operatorname{dist}(y_3, B_3) + r(B_3) \le \frac{1}{10} \min\big(\operatorname{dist}(y_3, \mathbb{R}^2_-), r(B_2)\big). \tag{4.19}$$

In the case that B_2 is below B_1 (which happens if B_1 is above S), using that $\Gamma = \partial\Omega^-$, we can choose the ball B_3 so that $B_3 \subset \Omega^-$ satisfies also (4.19).

In any case, let C_3 be the circumference centered at x passing through the center of B_3. Then it follows that C_3 intersects B_1, B_2, B_3. Observe that, in either case $B_1, B_2, B_3 \subset \mathbb{R}^2_+$.

In the first case, there is an arc in C_3 whose end-points belong respectively to B_1, B_3 (which are contained in Ω^+), passes through B_2, and its length is smaller than $\mathcal{H}^1(C_3)/2$, due to the fact that its end-points belong to $\mathbb{R}^2_+$. By the Key Property, this arc is contained in $\overline{\Omega^+}$, which is a contradiction because $B_2 \subset \Omega^-$. In the second case we deduce that there is an arc in C_3 that joins B_2 and B_3 (which are contained in Ω^-) and passes through B_1, with length smaller than $\mathcal{H}^1(C_3)/2$. By the Key Property, the arc is contained in $\overline{\Omega^-}$. This is again contradiction, because $B_1 \subset \Omega^+$. Hence, the curvature of S at x is zero.

We now appeal to the following simple fact.

LEMMA 4.21. *If a real analytic variety $Z \subset \mathbb{R}^2$ contains a segment S, then it also contains the line L that supports the segment.*

PROOF OF 4.21. By a suitable translation and rotation we can assume that the line L supporting S coincides with the horizontal axis of $\mathbb{R}^2$. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a real analytic function such that $Z = \Phi^{-1}(0)$. Then, the function defined by $\phi(x_1, x_2) = \Phi(x_1, 0)$ is real analytic, and it vanishes in the interior of the set $S \times \mathbb{R}$ and thus it vanishes identically in $\mathbb{R}^2$. That is, $\Phi(x_1, 0) = 0$ for all $x_1 \in \mathbb{R}$, or, in other words, $L \subset \Phi^{-1}(0) = Z$. □

Returning to the proof of Lemma 4.20, Lemma 4.21 shows that Γ contains a line L such that $\mu(L \cap \frac{1}{4}B) > 0$.

Our next objective consists of showing that $\Gamma \cap B \subset L \cap B$, which will complete the proof of the lemma. Again, without loss of generality, suppose that L is the horizontal axis.

Suppose that $B \cap \mathbb{R}^2_+ \cap \Omega^+ \neq \varnothing$. We intend to show that then $B \cap \mathbb{R}^2_+ \subset \Omega^+$.

For $x \in L$, consider the following semicircular extension of $\Omega^+ \cap B(x, 3r(B)) \cap \mathbb{R}^2_+ \cap$ with respect to the center x:

$$U_x^+ = \bigcup_{r \in (0,3r(B)):\partial B(x,r)\cap\Omega^+\cap\mathbb{R}^2_+ \neq \varnothing} \big(\partial B(x,r) \cap \mathbb{R}^2_+\big). \tag{4.20}$$

Observe that U_x^+ is also an open set.

CLAIM 7. *If $x \in \operatorname{supp}(\mu) \cap B(x, 3r(B)) \cap \mathbb{R}^2_+$, then*

$$\Omega^+ \cap B(x, 3r(B)) = U_x^+.$$

PROOF OF CLAIM 7. The arguments we use are similar to those required to show that the curve S had vanishing curvature. We need to show that $U_x^+ \subset \Omega^+$ (recall $L \subset \partial\Omega^+$). Assuming otherwise, there exists some point $y \in U_x^+ \cap \overline{\Omega^-}$. By connectivity, then we deduce that there exists some $r \in (0, 3r(B))$ such that

$$\partial B(x,r) \cap \Omega^+ \cap \mathbb{R}^2_+ \neq \varnothing \quad \text{and} \quad \partial B(x,r) \cap \Gamma \cap \mathbb{R}^2_+ \neq \varnothing.$$

Because of the existence of some point $y' \in \partial B(x,r) \cap \Gamma \cap \mathbb{R}^2_+$, the fact that $\Gamma = \partial\Omega^-$, and the openness of U_x^+, we deduce that there exists some ball $B_1' \subset U_x^+ \cap \Omega^-$. Let C_1' be the circumference centered at x passing through the center of B_1'. Choose one of the two points $z \in C_1' \cap L$ so that the *shortest* arc in C_1' that joints z to B_1' intersects Ω^+. See Figure 3 below.

Again by the fact that $\Gamma = \partial\Omega^-$, there exists some ball $B_2' \subset \Omega^-$ such that

$$r(B_2') + \operatorname{dist}(z, B_2') \leq \frac{1}{100} \min\big(r(B_1'), \operatorname{dist}(B_1', L)\big).$$

Let C_2' be the circumference centered at x passing through the center of B_2'. It is easy to check that there is an arc $I' \subset C_2'$ whose end-points belong respectively to B_1' and B_2', such that it intersects Ω^+, and moreover has length smaller that $\frac{1}{2}\mathcal{H}^1(C_2')$. Since B_1' and B_2' are contained in Ω^-, the whole I' is contained in $\overline{\Omega^-}$ by the Key Property, which contradicts the fact that $I' \cap \Omega^+ \neq \varnothing$. □

Recall that we are assuming that $B \cap \mathbb{R}^2_+ \cap \Omega^+ \neq \varnothing$ and we want to show that then $B \cap \mathbb{R}^2_+ \subset \Omega^+$. Suppose that this not the case. Of course, this implies that if $x_B \in \operatorname{supp}(\mu) \cap L$ is the centre of B, then $B(x_B, 2r(B)) \cap \mathbb{R}^2_+ \not\subset \Omega^+$. Let V be a connected component of $\Omega^+ \cap B(x_B, 2r(B)) \cap \mathbb{R}^2_+$. Since, by Claim 7, V coincides with its semicircular extension centered at x_B, it is of the form

$$V = A(x_B, s_1, s_2) \cap \mathbb{R}^2_+ \quad \text{or} \quad V = B(x_B, s_1) \cap \mathbb{R}^2_+,$$

with $s_1 < 2r(B)$ in any case (because $V \neq B(x_B, 2r(B)) \cap \mathbb{R}^2_+$ by assumption). Let $x' \in \frac{1}{10}B \cap L \cap \operatorname{supp}\mu$, $x' \neq x_B$ (the existence of x' is an immediate consequence of the linear growth of μ). By Claim 7, the semicircular extension $U_{x'}$ centred at x' is also contained in Ω^+, but

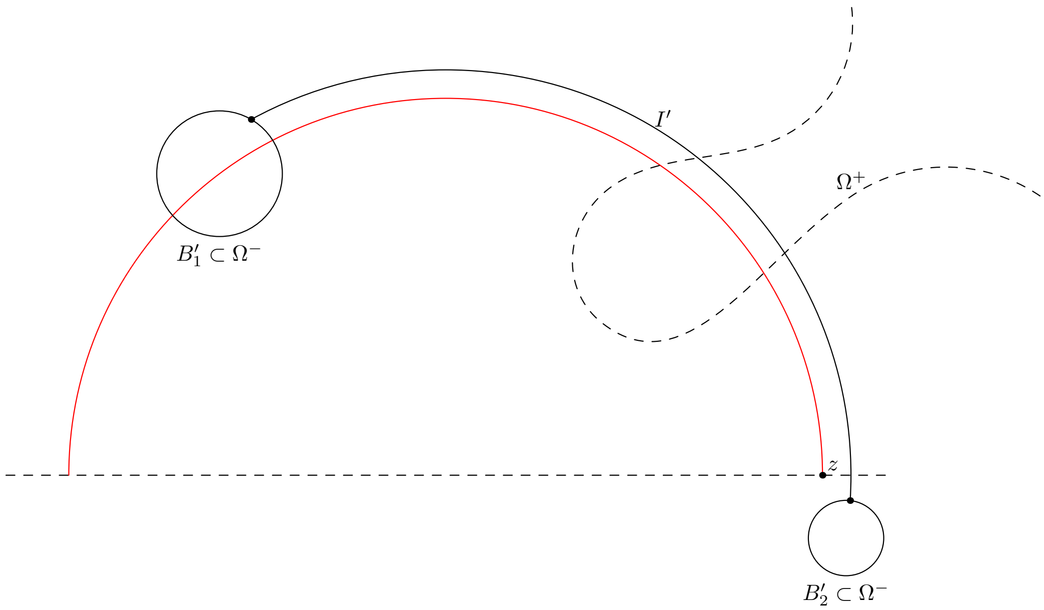

FIGURE 3. The figure depicts geometric set up in the proof of Claim 7. In particular, observe the arc I' with end-points belonging to Ω^- that intersects Ω^+.

then

$$\Omega_+ \supset U_{x'} \supset \bigcup_{r \in (0,3r(B)) : \partial B(x',r) \cap \mathbb{R}^2_+ \cap V \neq \varnothing} (\partial B(x',r) \cap \mathbb{R}^2_+) \supset \partial B(x_B, s_1) \cap \mathbb{R}^2_+,$$

which contradicts the definition of V as a connected component of Ω^+.

We have now verified that, if $B \cap \mathbb{R}^2_+ \cap \Omega^+ \neq \varnothing$ then $B \cap \mathbb{R}^2_+ \subset \Omega^+$. But by completely analogous arguments, we see that if $B \cap \mathbb{R}^2_+ \cap \Omega^- \neq \varnothing$ then $B \cap \mathbb{R}^2_+ \subset \Omega^-$, and one can interchange the upper half plane with the lower half plane. We therefore conclude that $B \cap \partial \Omega^+ \subset L$, and the proof of the Lemma 4.20 is complete. $\square$

Proof of Lemma 4.18. By renormalizing it suffices to prove the lemma for the ball $B_0 := B(0,1)$. We argue by contradiction: then there exists an $\epsilon > 0$ such that for all $k \in \mathbb{N}$, there exists a 2-sided c-corkscrew open set Ω_k^+ with $\Gamma_k := \partial \Omega_k^+$ containing 0 supporting a measure μ_k with C_0-linear growth with $\mu_k(B_0) > c_0$, so that we have

$$\int_{7B_0} \int_0^7 \left(\epsilon_k(x,r)^2 + \alpha_k^+(x,r)^2\right) \frac{dr}{r} \, d\mu_k(x) \le \frac{1}{k} \, \mu_k(7B_0), \tag{4.21}$$

and

$$\beta_{\infty,\Gamma_k}(B_0) > \epsilon.$$

Here we denote by $\epsilon_k(x,r)$ and $\alpha_k^+(x,r)$ the coefficients $\epsilon(x,r)$ and $\alpha^+(x,r)$ associated with Ω_k^+.

Observe that the condition $\mu_k(B_0) \approx 1$ and the linear growth of μ_k imply that $\operatorname{diam}(\Omega_k^+) \geq \operatorname{diam}(\Gamma_k) \geq \operatorname{diam}(B_0 \cap \operatorname{supp} \mu_k) \gtrsim 1$. Therefore, passing to a subsequence (which we relabel) if necessary, we may apply Lemma 4.19 to find a 2-sided c-corkscrew open sets $\Omega^\pm$ such that $\Gamma = \partial \Omega^\pm$ and

$$\lim_{k \to \infty} \Omega_k^\pm = \Omega^\pm \text{ and } \lim_{k \to \infty} \Gamma_k = \Gamma \text{ locally as } k \to \infty.$$

This implies that $\beta_{\infty,\Gamma}(\overline{B_0}) \geq \epsilon$.

Next, Lemma 4.10 ensures that, by passing to a further subsequence if necessary, we may assume that the measures μ_k converge weakly to a measure μ, supported on Γ, with C_0-linear growth and $\mu(\overline{B_0}) \geq c_0$.

We now apply Lemma 4.17. Therefore, there is an analytic variety Z such that $7B_0 \cap \operatorname{supp}(\mu) \subset Z \subset \Gamma$, and for every $x \in 7B_0 \cap \operatorname{supp}(\mu)$ and $r \in (0,7)$

(4.22) there are complementary semicircumferences $C^+(x,r), C^-(x,r)$ centered at x with radius r satisfying $C^\pm(x,r) \subset \Omega^\pm$

Since $\mu(\overline{B_0}) \geq c_0$, we can now find a ball B' centred on $\operatorname{supp}(\mu) \cap \overline{B_0}$ such that $7B_0 \supset B' \supset \overline{B_0}$ such that $\operatorname{supp}(\mu) \cap B' \subset Z$ and (4.22) holds for every $x \in \operatorname{supp}(\mu) \cap B'$ and $r \in (0, 3r(B'))$. We now apply Lemma 4.20 with the ball B' to conclude that $\Gamma_\infty \cap B'$ (and so $\Gamma \cap \overline{B_0}$) is a segment. This, however, contradicts the fact that $\beta_{\infty,\Gamma}(\overline{B_0}) \geq \epsilon$. □

8. The case of Jordan domains

In this section we shall prove Main Lemma 4.5 in the case of a Jordan domain, which we restate for the benefit of the reader.

Lemma 4.22. *Let $\Omega^+ \subset \mathbb{R}^2$ be a Jordan domain, let $\Gamma = \partial\Omega^+$, and let μ be a measure with C_0-linear growth supported on Γ. Let B be a ball centered at Γ such that*

$$\mu(B) \geq c_0 r(B),$$

for some $c_0 \in (0, C_0)$. Given any $\epsilon > 0$, there exists $\delta > 0$ (depending on C_0, c_0, ϵ) such that if

$$\int_{7B}\int_0^{7r(B)} \left(\epsilon(x,r)^2 + \alpha^+(x,r)^2\right) d\mu(x)\frac{dr}{r} \leq \delta\, \mu(7B),$$

then

$$\beta_{\infty,\Gamma}(B) \leq \epsilon.$$

The first auxiliary result we need is the following, which states that, at points where the Carleson square function is sufficiently small, we may find corkscrew balls.

Lemma 4.23. *Let $\Omega^+ \subset \mathbb{R}^2$ be a Jordan domain. Let $x \in \Gamma = \partial\Omega^+$, $r > 0$, and $x' \in \Gamma \cap \partial B(x,r)$. Suppose that*

$$\int_0^{2r} \left(\epsilon(x,t)^2 + \epsilon(x',t)^2\right) \frac{dt}{t} \leq \delta,$$

for some $\delta > 0$. If δ is small enough, then there are two balls $B^\pm \subset B(x,r) \cap \Omega^\pm$ such that $r(B^+) \approx r(B^-) \approx r$, where the implicit constants are absolute.

Proof. Without loss of generality, we may assume that $x = 0$, $r = 1$, and x' lies on the horizontal axis. It will be convenient to work with rectangles in polar coordinates. For intervals $I \subset (0,\infty)$ and $P \subset [-\pi,\pi]$, define

$$\mathrm{X}(I,P) = \{se^{i\theta} : s \in I, \theta \in P\}.$$

We call such a set a *polar rectangle*. Note that, if $I \subset [0,1]$, then we can inscribe a ball inside $\mathrm{X}(I,P)$ with radius a constant multiple of $\ell(I)\ell(P) \approx \sqrt{m_2(\mathrm{X}(I,P))}$.

We begin with a claim.

Claim 8. *There is an absolute constant $c > 0$ such that the following holds. For intervals $I \subset [1/2,1]$ and $P \subset [\pi/4, 3\pi/4] \cup [-3\pi/4, -\pi/4]$, if δ is sufficiently small (depending on $\ell(I)$ and $\ell(P)$), then there exists a polar rectangle $\mathrm{X}' \subset \mathrm{X}(I,P)$ such that $m_2(\mathrm{X}') \geq c m_2(\mathrm{X})$ and either $\mathrm{X}' \subset \Omega^+$ or $\mathrm{X}' \subset \Omega^-$.*

Let us first show how to prove the lemma using Claim 8. First, take $I = [1/2,1]$ and $P = [\pi/4, 3\pi/4]$. Then we get a polar rectangle $\mathrm{X}' = \mathrm{X}(I',P') \subset \mathrm{X}(I,P)$ with $m_2(X') \gtrsim 1$, and such that $\mathrm{X}' \subset \Omega^\pm$, provided δ is small enough. For definiteness let us assume that $\mathrm{X}' \subset \Omega^+$. We then apply the claim again with I replaced by I' and P replaced by $-(\frac{1}{3}P') = \{-\theta : \theta \in \frac{1}{3}P'\}$ (here for an interval P, aP is the concentric interval of sidelength $a\ell(P)$). As long as δ is small enough, there is a polar rectangle $\mathrm{X}'' = \mathrm{X}(I'',P'') \subset \mathrm{X}(I', -(\frac{1}{3}P'))$ with $\mathrm{X}'' \subset \Omega^\pm$, and $m_2(\mathrm{X}'') \gtrsim 1$. We need to verify that $\mathrm{X}'' \subset \Omega^-$.

However, if $\mathrm{X}'' \subset \Omega^+$, then we would have that every circumference $C(0,s)$, with $s \in I''$, has its intersection with X'' or X' contained in Ω^+. But $C(0,s)\backslash(\mathrm{X}' \cup \mathrm{X}'')$ is comprised of two arcs with length at most equal to $(\pi - \ell(P''))s$. Thus $\epsilon(0,s) \gtrsim 1$ for all $s \in I''$, whence $\int_0^1 \epsilon(0,s)\frac{ds}{s} \gtrsim 1$. We have therefore arrived at a contradiction if δ is sufficiently small. $\square$

We now return to verify the claim.

Proof of Claim 8. We may assume that $X = X(I,P) \subset \mathbb{R}^2_+$, the upper half-plane (i.e., $P \subset [\pi/4, 3\pi/4]$). First split $X(I,P)$ into 1000 polar rectangles $\mathrm{X}_j = \mathrm{X}(I,P_j)$ with $\ell(P_j) = \frac{1}{1000}\ell(P)$. Write $I = [r_1, r_2]$. Fix $\varkappa > 0$, and consider the circumferences

$$C_s = \partial B(0,s) \text{ for } s \in ((1-\varkappa)r_2 + \varkappa r_1, r_2).$$

If δ is sufficiently small, 99% of these circumferences intersect Γ in at most 4 of the polar rectangles X_j. In this case, we call C_s *good*.

Next, for fixed $\varkappa > 0$ and for each polar rectangle X_j, set

$$S_\varkappa(j) := \{s \in [(1-\varkappa)r_2 + \varkappa r_1, r_2] \,|\, C_s \text{ is good and } \Gamma \cap X_j \cap C_s \neq \varnothing\}.$$

Then put $m_j := \mathcal{H}^1(S_\varkappa(j))$. The number m_j, then, describes the amount of good circumferences which hit Γ at X_j. Fubini's theorem yields

$$\begin{aligned}\sum_j m_j &= \sum_j \int \mathbb{1}_{S_\varkappa(j)}(s)\,ds = \sum \int \mathbb{1}_{S_k(j)\times\{j:\exists s\in I_\varkappa \text{ s.t. } C_s \text{ good, } \Gamma\cap X_j\cap C_s\neq\varnothing\}}(s,j)\,ds\\ &= \int \sum \mathbb{1}_{S_k(j)\times\{j:\exists s\in I_\varkappa \text{ s.t. } C_s \text{ good, } \Gamma\cap X_j\cap C_s\neq\varnothing\}}(s,j)\,ds\\ &\leq \int_{I_\varkappa} 4ds = 4\varkappa(r_2 - r_1).\end{aligned}$$

In particular, this implies that there exists a j_0 with $m_{j_0} \leq \frac{4\varkappa}{1000}(r_2 - r_1)$.

Consequently,

$$\begin{aligned}&\mathcal{H}^1\big(\{s \in [(1-\varkappa)r_2 + \varkappa r_1, r_2] : C_s \cap \mathrm{X}_{j_0} \cap \Gamma \neq \varnothing\}\big)\\ &\quad\leq m_{j_0} + \mathcal{H}^1(\{s \in ((1-\varkappa)r_2 + \varkappa r_1, r_2) : C_s \text{ is not good}\})\\ &\quad\leq \frac{4\varkappa}{1000}(r_2 - r_1) + \frac{1}{100}\varkappa(r_2 - r_1) \leq \frac{1}{50}\varkappa(r_2 - r_1),\end{aligned}$$

and we conclude that at most only 2% of the circumferences C_s, $s \in ((1-\varkappa)r_2 + \varkappa r_1, r_2)$, intersect Γ in $\mathrm{X}_{j_0} = \mathrm{X}(I, P_{j_0})$.

Using the pigeonhole principle, we infer that we can find three pairwise disjoint intervals I_1, I_2 and I_3 in $[(1-\varkappa)r_2 + \varkappa r_1, r_2]$, such that

- $\ell(I_k) \gtrsim \varkappa(r_2 - r_1)$,
- $\mathrm{dist}(I_j, I_k) \gtrsim \varkappa(r_2 - r_1)$ if $j \neq k$, and
- $C(0,s) \cap \mathrm{X}_{j_0} \cap \Gamma = \varnothing$ whenever $s \in \partial I_k$ for $k = 1, 2, 3$.

Consider the three polar rectangles $\mathrm{X}_{j_0,k} = \mathrm{X}(I_k, P_{j_0})$, which certainly contain $\widetilde{\mathrm{X}}_{j_0,k} = \mathrm{X}(I_k, \widetilde{P}_{j_0})$ with $\widetilde{P}_{j_0} = \frac{1}{10}P_{j_0}$. We will show that one of the rectangles $\widetilde{\mathrm{X}}_{j_0,k}$, for some $k = 1, 2, 3$, does not intersect Γ.

Let us write $\Gamma = \gamma([0,1])$ with $\gamma(0) = 0 = \gamma(1)$. First suppose $\Gamma \cap \widetilde{\mathrm{X}}_{j_0,k} \neq \varnothing$ for some $k \in \{1,2,3\}$. If we consider u_0 such that $\gamma(u_0) \in \widetilde{\mathrm{X}}_{j_0,k}$ and $u_1 = \max\{u : \gamma([u_0, u]) \subset \mathrm{X}_{j_0,k}\}$, then since $C(0,s) \cap \mathrm{X}_{j_0,k} \cap \Gamma = \varnothing$ for $s \in \partial I_k$, and $0 = \gamma(0) = \gamma(1) \notin \mathrm{X}_{j_0,k}$, we must have that $\gamma(u_1) \in \{se^{i\theta} : s \in I_k, \theta \in \partial P_{j_0}\}$. We say that Γ goes to the right (left resp.) if $\gamma(u_1)$ lies on the right (left) side boundary of $\mathrm{X}_{j_0,k}$. Assuming that $\widetilde{\mathrm{X}}_{j_0,k} \cap \Gamma \neq \varnothing$ for every $k = 1, 2, 3$, we therefore see that Γ must go to one direction (either left or right) in two of the rectangles, say X_{j_0,k_1} and X_{j_0,k_2} – for definiteness let us say the direction is right (analogues arguments handle the other case).

If we fix $\varkappa = 10^{-6}\ell(P_j)$, say, then there is an interval J with $\ell(J) \gtrsim \varkappa\ell(I)$ so that for every $s \in J$, the circumference $C(x', s)$ crosses X_{j_0,k_1} to the right of $\widetilde{\mathrm{X}}_{j_0,k_1}$ and also crosses X_{j_0,k_2} to the right of $\widetilde{\mathrm{X}}_{j_0,k_2}$. Therefore, insofar as Γ goes to the right in both X_{j_0,k_1} and X_{j_0,k_2}, a circumference $C(x', s)$ with $s \in J$ intersects Γ in the well-separated polar rectangles X_{j_0,k_1} and X_{j_0,k_2} (see Figure 4), and so $\epsilon(x', s) \gtrsim_{\varkappa,\ell(I)} 1$. But then $\int_0^2 \epsilon(x', s)^2\frac{ds}{s} \gtrsim_{\varkappa,\ell(I)} 1$. If δ is small enough then we have reached a contradiction. $\square$

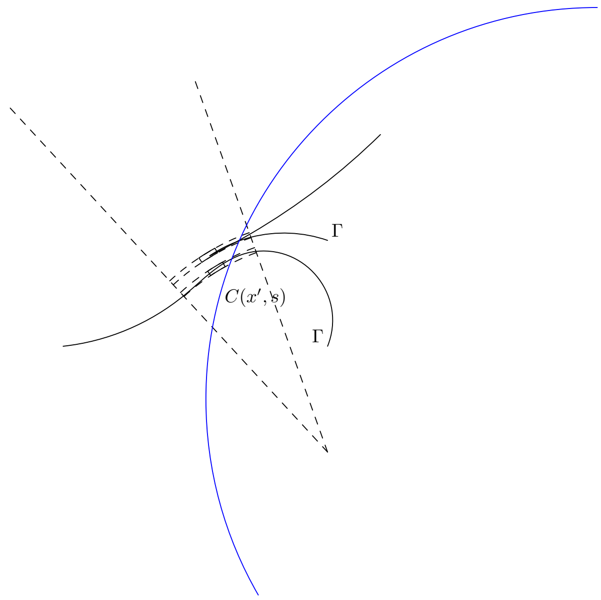

FIGURE 4. The figure depicts a circumference $C(x', s)$ crossing X_{j_0,k_1} to the right of $\widetilde{\mathrm{X}}_{j_0,k_1}$ and crossing X_{j_0,k_2} to the right of $\widetilde{\mathrm{X}}_{j_0,k_2}$.

We begin by reviewing Lemma 4.14 in the context of a sequences of Jordan domains.

LEMMA 4.24. *Let $\{\Omega_j^+\}_j$ be a sequence of Jordan domains in the plane such that $0 \in \partial\Omega_j^+$. Let $\Omega_j^- = \mathbb{R}^2 \setminus \overline{\Omega_j^+}$ and $\Gamma_j = \partial\Omega_j^+$. Then the following holds:*

(1) There is a subsequence of domains $\Omega_{j_k}^\pm$ and there are closed sets G^+, G^-, G_0 such that

$$\overline{\Omega_{j_k}^\pm} \to G^\pm \quad \text{and} \quad \Gamma_{j_k} \to G_0 \quad \text{locally.}$$

(2) The limit sets G^+, G^-, G_0 satisfy

$$G^+ \cup G^- = \mathbb{R}^2, \qquad G^+ \cap G^- = G_0.$$

In particular, $G^+ \setminus G_0$ and $G^- \setminus G_0$ are open.

PROOF. The existence of the locally convergent subsequences follows from Lemma 4.13, the property (2) is then a consequence of Lemma 4.14. □

We remark that, in the above situation, G_0 need not coincide with ∂G^+ or ∂G^-. Further, G_0 may have non-empty interior, and $G^\pm \setminus G_0$ may be empty.

Our next lemma reviews the basic convergence result Lemma 4.17, also taking into account Lemma 4.23.

LEMMA 4.25. *Let $\{\Omega_j^+\}_j$ be a sequence of Jordan domains in the plane which intersect some ball B_0. Let $\Omega_j^- = \mathbb{R}^2 \setminus \overline{\Omega_j^+}$ and $\Gamma_j = \partial\Omega_j^+$. Suppose $G^\pm$ and G_0 are closed sets with*

$$\overline{\Omega_j^\pm} \to G^\pm \text{ and } \Gamma_j \to G_0 \text{ locally as } j \to \infty.$$

Suppose μ_j are measures supported on Γ_j with C_0-linear growth that converge weakly to a measure μ_0 satisfying $\mu_0(\overline{B_0}) \geq c_0 r(B_0)$. Suppose that (4.13) holds, i.e.

$$\int_{7B_0} \int_0^{7r(B_0)} \alpha_j^+(x,r)^2 \, \frac{dr}{r} \, d\mu_j(x) \leq \frac{1}{j} \, \mu_j(7B_0),$$

where α_j^+ are the coefficients α^+ associated with Ω_j^+, and (4.14) holds, i.e.,

$$\int_{7B_0}\int_0^{7r(B_0)} \epsilon_j(x,r)^2\,\frac{dr}{r}\,d\mu_j(x) \le \frac{1}{j}\,\mu_j(7B_0),$$

where $\epsilon_j(\cdot,\cdot)$ are the coefficients $\epsilon(\cdot,\cdot)$ associated with Ω_j^+. Then

(1) *there is an analytic variety Z with $7B_0 \cap \operatorname{supp}(\mu_0) \subset Z \subset G_0$.*
(2) *for all $x \in 7B_0 \cap \operatorname{supp}\mu_0$ and all $r \in (0, 7r(B_0))$ there is a pair of admissible semicircumferences which are contained in $\partial B(x,r)$.*
(3) *for every $M > 0$, there exists a constant $c(M) > 0$ such that whenever $x \in B_0 \cap \operatorname{supp}(\mu_0)$ and $r \in (0, r(B_0))$ are such that $\mu_0(B(x,r)) \ge r/M$, then there are two balls $B^\pm \subset B(x,2r) \cap G^\pm \setminus G_0$ with $r(B^\pm) \ge c(M)r$.*

Proof. The proof of the first two statements are precisely those of Lemma 4.17. The third assertion is proved by passing to the limit in the result in Lemma 4.23. Indeed, fix $r' = \frac{r}{3MC_0}$. Then for sufficiently large j (and using linear growth), $\mu_j(B(x,r)\backslash B(x,r')) \ge \frac{r}{2M}$. Since $x \in \operatorname{supp}(\mu)$, for any $s \in (0, \frac{r'}{2})$ we have $\liminf_{j\to\infty} \mu_j(B(x,s)) > 0$, whence

$$\frac{1}{\mu_j(B(0,s))}\int_{B(x,s)}\int_0^{7r} \epsilon_j^+(y,r)^2\,\frac{dr}{r}\,d\mu_j(y) \le \frac{1}{j\mu_j(B(0,s))}\,\mu_j(7B_0) \to 0 \text{ as } j\to\infty.$$

Consequently, for $\delta > 0$ as in Lemma 4.23, and for sufficiently large j we can find $z_j \in B(x,s) \cap \operatorname{supp}(\mu_j)$ with $\int_0^{7r} \epsilon_j^+(z_j,r)^2\,\frac{dr}{r} < \delta$. But now, as $j\to\infty$,

$$\frac{1}{\mu_j(B(x,r)\backslash B(x,r'))}\int_{B(x,r)\backslash B(x,r')}\int_0^{7r} \epsilon_j^+(y,r)^2\,\frac{dr}{r}\,d\mu_j(y) \le \frac{2M}{jr}\mu_j(7B_0) \to 0,$$

so for large j we can find $z_j' \in \operatorname{supp}(\mu_j) \cap B(x,r)\backslash B(x,r')$ with $\int_0^{7r} \epsilon_j^+(z_j',r)^2\,\frac{dr}{r} < \delta$. Notice that $z_j' \in \partial B(z_j, t_j)$ with $t_j \in (r'/2, \frac{3}{2}r)$. We apply Lemma 4.23 with the points z_j and z_j' and radius t_j (note that $2t_j \le 7r$) to find balls $B_j^\pm \in \Omega_j^\pm \cap B(z_j, \frac{3}{2}r)$ such that $r(B_j^\pm) \gtrsim \frac{r}{M}$. If s is small enough, $B_j^\pm \subset \overline{B(x,3r/2)}$ and we may pass to a subsequence $B_{j_k}^\pm$ which converge in Hausdorff distance to balls[3] $B^\pm \subset G^\pm$ with $B^\pm \subset B(x,2r)$. But then if $y \in B^\pm$ (say $y \in B^+$ for definiteness), then a neighbourhood of y is contained in $B_{j_k}^+$ for sufficiently large k, and so $\liminf_k \operatorname{dist}(y, \Gamma_{j_k}) > 0$, which ensures that $y \in G^+\backslash G_0$. Thus $B^\pm \subset G^\pm\backslash G_0$. $\square$

Our next result is an analogue of Lemma 4.20. The reader should notice that the conclusion is weaker. This is due to the fact that we only can infer anything about the structure of the boundary set G_0 at points where μ has lots of mass (via property (3) of Lemma 4.25).

Lemma 4.26. *Suppose G^+, G^- and G_0 are three closed sets satisfying $G^+ \cup G^- = \mathbb{R}^2$ and $G^+ \cap G^- = G_0$. Suppose that μ_0 is a measure with C_0-linear growth, let $B_0 \subset \mathbb{R}^2$ be some ball centered on $\operatorname{spt}(\mu_0)$ such that $\mu(B_0) > 0$ and assume that there is a real analytic variety Z with $\operatorname{spt}(\mu_0) \cap B_0 \subset Z \subset G_0$. Suppose moreover, that*

(1) *for every $M > 0$, there exists a constant $c(M) > 0$ such that whenever $x \in B_0 \cap \operatorname{supp}(\mu_0)$ and $r \in (0, r(B_0))$ are such that $\mu_0(B(x,r)) \ge r/M$, then there are two balls $B^\pm \subset B(x,2r) \cap G^\pm \setminus G_0$ with $r(B^\pm) \ge c(M)r$.*
(2) *for every $x \in 2B_0 \cap \operatorname{spt}(\mu_0)$ and $r \in (0, 2r(B_0))$ there exists two complementary semicircumferences $C^+(x,r)$, $C^-(x,r)$ with radius r and center x such that*

$$C^+(x,r) \subset G^+ \quad \text{and} \quad C^-(x,r) \subset G^-.$$

Then there is some line L such that

$$L \subset G_0 \quad \text{and} \quad \mu_0(B_0 \cap L) > 0.$$

There is some natural repetition in the proof of Lemma 4.26 and Lemma 4.20, but since the proofs are also quite substantially different, and some readers may want to only consider the case of Jordan domains, we repeat all the relevant details here.

[3] That $B^\pm \subset G^\pm$ is a consequence of Lemma 4.12,(1).

PROOF. Since μ_0 is non-zero and has linear growth, it is clear that $\mathcal{H}^1(Z) \geq \mathcal{H}^1(\operatorname{spt}(\mu_0) \cap B_0) > 0$. Together with the fact that $Z \neq \mathbb{R}^2$ (which follows from property (1), since $\mu(B_0) > 0$), this implies that there exists some analytic arc S such that $\mu_0(S \cap B_0) > 0$ (which implies that $\mathcal{H}^1(S \cap \operatorname{spt}(\mu_0) \cap B_0) > 0$, because of the linear growth of μ_0).

We claim that S is a segment. To prove this it suffices to show that S has vanishing curvature in a set positive measure μ_0. Indeed, since this set has positive length and the curvature of a real analytic arc is locally a real analytic function (with respect the arc-length parametrization from an interval), this implies that the curvature vanishes on the whole arc S . Thus S is a segment.

To show that the curvature of S vanishes in some set of positive measure μ_0, we will again use the

KEY PROPERTY. *Given $x \in B \cap \operatorname{supp} \mu_0$ and $r \in (0, 3r(B_0))$, let $I \subset \partial B(x,r)$ be an arc such that $\mathcal{H}^1(I) < \pi r$ whose end-points belong both to $G^+ \setminus G_0$. Then $I \subset G^+$. The analogous statement holds replacing G^+ by G^-.*

We verify the Key Property as follows: if I is an arc as above, then its end-points x_1, x_2 do not belong to G^- (because they belong to $G^+ \setminus G_0$). Hence, if (C^+, C^-) is a pair of complementary semicircumferences at x with radius r satisfying $C^{\pm} \subset G^{\pm}$, we have that $x_1, x_2 \in C^+$, and thus either I or $\partial B(x,r) \setminus I$ is contained in C^+. The latter cannot happen since $\mathcal{H}^1(\partial B(x,r) \setminus I) > \pi r = \mathcal{H}^1(C^+)$, and so we have $I \subset C^+ \subset G^+$.

We are ready now to show that the curvature of S vanishes at some set of positive measure μ_0. We consider some set $F \subset S$ such that $\mathcal{H}^1(F) > 0$ and $\mu_0|_F = h\,\mathcal{H}^1|_F$ for some function $h \approx 1$ (with the implicit constant possibly depending on S, F, and other parameters). Without loss of generality we assume that $x_0 = 0$ is a density point of F and that the tangent to S at x_0 is the horizontal axis. Aiming for a contradiction, suppose that S is strictly convex at x_0 (i.e., if S equals the graph of the real analytic function $g : (-\delta, \delta) \to \mathbb{R}$ in a neighborhood of x_0, then $g''(0) > 0$).

Let z_1, z_2 be the two end-points of S, and let

$$d_0 = \frac{1}{2} \min_{i=1,2} \operatorname{dist}(x_0, z_i).$$

Let $r \in (0, d_0/2)$ be small enough so that $g''(\pi_H(x))$ is comparable to $g''(\pi_H(x_0))$ in $B(x_0, r) \cap S$, where π_H is the orthogonal projection on the horizontal axis. We will prove the following:

CLAIM 9. *There exist some $z \in B(x_0, r/10) \cap S \cap \operatorname{supp} \mu_0$ and some $r' \in (0, r/10)$ and an arc $I \subset \partial B(z, r')$ with $\mathcal{H}^1(I) < \pi r'$ such that either its end-points belong to $G^+ \setminus G_0$ and intersects $G^- \setminus G_0$, or its end-points belong to $G^- \setminus G_0$ and intersects $G^+ \setminus G_0$.*

The preceding claim asserts that the strict convexity of g at x_0 implies that the Key Property is violated. Hence S is a segment. Lemma 4.21 then ensures that Z also contains the line L that supports the segment, thereby completing the proof of the lemma (up to verification of the claim). □

PROOF OF CLAIM 9. Since x_0 is a density point of F in S, we can take some $t \in (0, r/10)$ such that $\mathcal{H}^1(F \cap B(x_0, t)) \geq (1 - \tau)\,\mathcal{H}^1(S \cap B(x_0, t))$, where $\tau \in (0, 10^{-3})$ is some small parameter to be fixed below.

Denote by L_H the horizontal axis and let $J = (-t/2, t/2) \subset L_H$, so that $S \cap B(x_0, t) \supset g(J)$. We will appeal to the following simple lemma.

LEMMA 4.27. *Fix $\varkappa > \tau$, and suppose that $\mathcal{F}$ is a finite family of pairwise disjoint intervals contained in $(-t/2, t/2)$ satisfying*

$$\sum_{T \in \mathcal{F}} \mathcal{H}^1(g(T)) \geq 3\varkappa \mathcal{H}^1(S \cap B(x_0, t)). \tag{4.23}$$

If $\mathcal{F}'$ denotes the subfamily of intervals $T \in \mathcal{F}$ satisfying $\mathcal{H}^1(g(T) \cap F) \geq \varkappa \mathcal{H}^1(g(T))$, then

$$\sum_{T \in \mathcal{F}'} \mathcal{H}^1(g(T)) \cap F) \geq \varkappa \mathcal{H}^1(S \cap B(x_0, t)).$$

Proof. Suppose the conclusion fails. Then insofar as $\mathcal{F}$ are pairwise disjoint intervals, and $\mathcal{H}^1(g(T) \cap F) < \varkappa \mathcal{H}^1(g(T))$ for $T \in \mathcal{F} \backslash \mathcal{F}'$, we have

$$\begin{aligned} \sum_{T \in \mathcal{F}} \mathcal{H}^1(g(T) \cap F) &= \sum_{T \in \mathcal{F}'} \mathcal{H}^1(g(T) \cap F) + \sum_{T \in \mathcal{F} \backslash \mathcal{F}'} \mathcal{H}^1(g(T) \cap F) \\ &< \varkappa \mathcal{H}^1(S \cap B(x_0, t) + \varkappa \sum_{T \in \mathcal{F} \backslash \mathcal{F}'} \mathcal{H}^1(g(T)) \\ &\leq \mathcal{H}^1(S \cap B(x_0, t)) + \varkappa \mathcal{H}^1(S \cap B(x_0, t)) = 2\varkappa \mathcal{H}^1(S \cap B(x_0, t)). \end{aligned} \tag{4.24}$$

But then, if $\varkappa$ is smaller than a universal constant,

$$\mathcal{H}^1(B(x_0, t) \cap F) \overset{(4.24)}{\leq} 2\varkappa \mathcal{H}^1(S \cap B(x_0, t)) + \mathcal{H}^1\Big(S \backslash \bigcup_{T \in \mathcal{F}} g(T)\Big) \overset{(4.23)}{\leq} (1 - \varkappa) \mathcal{H}^1(S \cap B(x_0, t)).$$

The right hand side is strictly smaller than $(1 - \tau)\mathcal{H}^1(S \cap B(x_0, t))$, which is our desired contradiction. $\square$

We split J into three intervals J_l, J_c, J_r (where l, c, r, stand for left, center, right) with disjoint interiors such that $\mathcal{H}^1(J_l) = \mathcal{H}^1(J_c) = \mathcal{H}^1(J_r) = \mathcal{H}^1(J)/3$. Next we split J_l into N intervals with disjoint interiors of the same length, and we take $N = c_1 \mathcal{H}^1(J)^{-1}$, with $c_1 \in (0, 1)$ to be chosen below (depending on $g''(0)$). We denote by $J_l^1, \ldots J_l^N$ this family of intervals. By standard arguments, we find a subfamily $\{J_l^k\}_{k \in K_l}$ of $\{J_l^1, \ldots, J_l^N\}$ such that the intervals $\{10 J_l^k\}_{k \in K_l}$ are pairwise disjoint and moreover

$$\sum_{k \in K_l} \mathcal{H}^1(g(J_l^k)) \gtrsim \mathcal{H}^1(g(J_l)) \gtrsim \mathcal{H}^1(S \cap B(x_0, t)). \tag{4.25}$$

Note that (4.25) implies (4.23) (with a choice of $\varkappa$ depending on the implicit constants). Thus we may apply Lemma 4.27, with $\varkappa$ some absolute constant (provided τ is small enough), to find a subfamily $\{J_l^k\}_{k \in H_l} \subset \{J_l^k\}_{k \in K_l}$ of the intervals J_l^k such that

$$\mathcal{H}^1(F \cap g(J_l^k)) \approx \mathcal{H}^1(g(J_l^k)), \tag{4.26}$$

and

$$\sum_{k \in H_l} \mathcal{H}^1(F \cap g(J_l^k)) \gtrsim \mathcal{H}^1(S \cap B(x_0, t)). \tag{4.27}$$

Next, note that the condition (4.26) ensures that we can apply property (1) from Lemma 4.26 for each $k \in H_l$ to find a ball B_k^+ satisfying

$$B_k^+ \subset U_{\ell(J_l^k)}(g(J_l^k)) \cap (G^+ \setminus G_0), \qquad \text{with } r(B_k^+) \approx \mathcal{H}^1(J_l^k),$$

where $U_\ell(A)$ stands for the ℓ-neighborhood of A. Observe that, by the strict convexity of S, we have $\operatorname{dist}(g(J_l^k), L_H) \approx \ell(J)^2$. On the other hand,

$$\operatorname{dist}(B_k^+, g(J_l^k)) \leq \ell(J_l^k) \leq \frac{\ell(J)}{N} = c_1 \ell(J)^2.$$

So if we choose c_1 small enough, then the balls B_k^+ are contained in $\mathbb{R}^2_+$ and far from L_H.

Next, let I_k, $k \in H_l$, be the projection of the balls B_k^+, $k \in H_l$, on the axis L_H. The intervals I_k, $k \in H_l$, are disjoint, and moreover,

$$\sum_{k \in H_l} \mathcal{H}^1(g(\tfrac{1}{10} I_j)) \gtrsim \sum_{k \in H_l} \mathcal{H}^1(g(I_j)) \overset{(4.27)}{\gtrsim} \mathcal{H}^1(S \cap B(x_0, t)).$$

Therefore, appealing to Lemma 4.27 once again, with $\varkappa$ some absolute constant, we find a family of indices $M_l \subset H_l$ such that

$$\mathcal{H}^1(F \cap g(\tfrac{1}{10} I_k)) \approx \mathcal{H}^1(g(\tfrac{1}{10} I_k)) \approx \ell(I_k) \text{ for every } k \in M_l \tag{4.28}$$

and

$$\sum_{k\in M_l} \mathcal{H}^1(F\cap g(\tfrac{1}{10}I_k)) \approx \mathcal{H}^1(S\cap B(x_0,t)).$$

Since (4.28) holds, we may apply property (1) for each $k \in M_l$ to find some ball B_k^- satisfying

$$B_k^- \subset U_{\frac{1}{10}\ell(I_k)}(g(\tfrac{1}{10}I_k)) \cap (G^- \setminus G_0), \qquad r(B_k^-) \approx \mathcal{H}^1(I_k).$$

Again, by the strict convexity of S, the balls B_k^- are contained in $\mathbb{R}^2_+$ and far away from L_H. Further, by construction the projection $\pi_H(B_k^-)$ is contained deep inside $\pi_H(B_k^+)$ for each $k \in M_l$. In fact, by shrinking the balls B_k^- if necessary, we can assume that

$$\pi_H(B_k^-) \subset \pi_H(\tfrac12 B_k^+) \quad \text{ for each } k\in M_l.$$

Now we denote

$$W_l = \bigcup_{k\in M_l} \pi(\tfrac12 B_k^-).$$

By the disjointness of the intervals $10J_l^k$, $k \in M_l$, the intervals $\pi(\frac12 B_k^-)$ are disjoint and we deduce that $\mathcal{H}^1(W_l) \approx \mathcal{H}^1(J) \approx t$.

Next we define an analogous family of balls $\{B_k^\pm\}_{k\in M_r}$ and a set W_r, replacing the left interval J_l by the right one J_r.

We claim that there is some $x \in J_c \cap \pi_H(F)$ such that

$$W_l \cap (2x - W_r) \neq \varnothing.$$

In fact, for an arbitrary point $y_r \in W_r$, the set $\{2x - y_r : x \in J_c \cap \pi_H(F)\}$ is of the form $I \setminus X$, where I is an interval of length $2\ell(J_c)$ which contains J_l and X is an exceptional set with length at most $2\mathcal{H}^1(J\setminus \pi_H(F)) \le c\mathcal{H}^1(g(J)\setminus F) \le c\tau\ell(J)$. So for τ small enough, $\{2x-y_r : x\in J_c\cap\pi_H(F)\}$ intersects W_l, since $\mathcal{H}^1(W_l)\approx \ell(J) \gg c\tau\ell(J)$.

The preceding argument shows that there exist $y_l \in W_l$, $y_r\in W_r$, and $x\in J_c\cap\pi_H(F)$ such that $y_l = 2x - y_r$, or equivalently,

$$x = \frac{y_l+y_r}{2}.$$

Observe that, in particular, this implies that $|x-y_l| = |x-y_r| \approx \ell(J)$.

Let $k \in M_l$ be such that $y_l \in \pi_H(\frac12 B_k^-)$ and $h\in M_r$ such that $y_r \in \pi_H(\frac12 B_h^-)$. By construction, there are points $y_l^\pm \in \frac12 B_k^\pm$ and $y_r^\pm \in \frac12 B_h^\pm$ such that

$$\pi_H(y_l^-) = \pi_H(y_l^+) = y_l \quad \text{ and } \quad \pi_H(y_r^-) = \pi_H(y_r^+) = y_r.$$

We claim that the circumference centered at $g(x)$ (observe that $g(x) \in F \subset S\cap \operatorname{supp}\mu_0$) with radius $|x-y_l|$ intersects the four balls $B_k^\pm$ and $B_h^\pm$. To see this, notice that

$$\big||g(x)-y_l^\pm| - |x-y_l|\big| \le (1-\cos\alpha^\pm)\,\ell(J) \lesssim (\alpha^\pm)^2\,\ell(J),$$

where $\alpha^\pm$ is the slope of the line passing through $g(x)$ and $y_l^\pm$, which satisfies $\alpha^\pm \lesssim \ell(J)$ (taking into account that $|x-y_l| = |x-y_r| \approx \ell(J)$ and the quadratic behavior of g close to $x_0 = 0$). Thus,

$$\big||g(x)-y_l^\pm| - |x-y_l|\big| \lesssim \ell(J)^3 \ll \ell(J)^2 \approx r(B_k^\pm).$$

Analogously,

$$\big||g(x)-y_r^\pm| - |x-y_l|\big| = \big||g(x)-y_r^\pm| - |x-y_r|\big| \lesssim \ell(J)^3 \ll \ell(J)^2 \approx r(B_h^\pm).$$

Hence the aforementioned circumference passes through the balls $B_k^\pm$, $B_h^\pm$.

Let $z = g(x)$ and $r' = |x-y_l|$. It is easy to check that there is an arc contained in the circumference $\partial B(z,r')$ satisfying the required properties in the claim. To see this, let H_z the open half-plane whose boundary equals the tangent to S at z and containing $S\setminus\{z\}$. It is easy to check that the four balls $B_k^\pm$, $B_h^\pm$ are contained in H_z, taking into account that $g''(\xi)\approx g''(0)$ in the whole interval J and choosing the constant c_1 above small enough if necessary. □

It would appear that Lemma 4.26 is the most we can extract out of the assumptions stated there, and in particular using only the existence of complementary pairs. To say more, we need to use the full strength of the admissible pairs property, which has a memory of the limiting sequence Ω_j^+ of Jordan domains.

Our goal will be to prove the following result.

LEMMA 4.28. *Suppose that Ω_j^+ is a sequence of Jordan domains such that there are closed sets G^+, G^- and G_0 such that, with $\Omega_j^- = \mathbb{R}^2 \setminus \overline{\Omega_j^+}$ and $\Gamma_j = \partial\Omega_j$,*

$$\lim_{j\to\infty} \overline{\Omega_j^\pm} = G^\pm, \quad \textit{and} \quad \lim_{j\to\infty} \overline{\Gamma_j} = G_0 \textit{ locally as } j \to \infty.$$

Suppose μ_0 is a measure supported in G_0 with C_0-linear growth, and B_0 is a ball satisfying

(1) there is a line $L \subset G_0$ with $\mu_0(B_0 \cap L) > 0$, and
(2) given any subsequence $\{\Omega_{j_k}^+\}_k$, and for every $x \in \operatorname{supp}(\mu_0)$ and $r \in (0, 3r_0)$, there exists a pair (S_1, S_2) that is admissible for the sequence of domains[4] $\{\Omega_{j_k}^+\}_k$ that is centered at x with radius $r > 0$.

Then $G_0 \cap B_0 \subset L$ and if H_1, H_2 are the two open half planes whose boundary is L, we have that either

$$H_1 \cap B_0 \subset G^+ \setminus G_0 \quad \textit{and} \quad H_2 \cap B_0 \subset G^- \setminus G_0,$$

or

$$H_1 \cap B_0 \subset G^- \setminus G_0 \quad \textit{and} \quad H_2 \cap B_0 \subset G^+ \setminus G_0.$$

Observe that if a sequence of Jordan domains Ω_j^+ satisfies the assumptions of Lemma 4.28, then so does any subsequence of the domains.

Now we need to introduce some additional notation. Given a pair complementary semicircumferences (S_1, S_2), we say that the two common end-points of S_1, S_2 are the end-points of the pair (S_1, S_2) and we denote the set of these end-points by $(S_1, S_2)_{\text{ep}}$.

LEMMA 4.29. *Under the notation and assumptions of Lemma 4.28, fix $x \in \operatorname{supp}\mu_0 \cap B_0$, and $y \in \partial B(x,r) \cap G_0$ for some $r \in (0, 2r(B_0))$. Fix $\ell_0 > 0$. Suppose that, for sufficiently large k, and given any subsequence of the domains, we can find sequences of pairs $(S_{1,r+1/k}, S_{2,r+1/k})$ and $(S_{1,r-1/k}, S_{2,r-1/k})$ which are admissible for the subsequence of domains, which are centred at x and have radii $r + 1/k$ and $r - 1/k$ respectively, and such that*

$$\liminf_{k\to\infty} \operatorname{dist}(y, (S_{1,r+1/k}, S_{2,r+1/k})_{\text{ep}}) \geq \ell_0 \textit{ and } \liminf_{k\to\infty} \operatorname{dist}(y, (S_{1,r-1/k}, S_{2,r-1/k})_{\text{ep}}) \geq \ell_0.$$

Then, there exists a subsequence of arcs $\gamma_{j_k} \subset \Gamma_{j_k}$ which converge in Hausdorff distance to an arc $I \subset \partial B(x,r)$ such that y is one of its end-points and $\mathcal{H}^1(I) \geq \ell_0/5$.

PROOF. Consider the sequence of radii $s_k = r(1 - \frac{1}{k})$ and $t_k = r(1 + \frac{1}{k})$. By assumption, we can find admissible pairs (S_{1,s_k}, S_{2,s_k}) centered at x with radii s_k satisfying,

$$\liminf_{k\to\infty} \operatorname{dist}(y, (S_{1,s_k}, S_{2,s_k})_{\text{ep}}) \geq \ell_0. \tag{4.29}$$

Consequently, with ϵ_k a decreasing sequence chosen much smaller than $1/k$, there is a subsequence j_k, and arcs $I_{s_k}^\pm \subset \Omega_{j_k}^\pm \cap \partial B(y_{j_k}, \widetilde{s}_k)$ such that

$$|x - y_{j_k}| \leq \epsilon_k, \quad |s_k - \widetilde{s}_k| \leq \epsilon_k, \text{ and } |\mathcal{H}^1(I_{s_k}^\pm) - \pi s_k| \leq \epsilon_k. \tag{4.30}$$

Also, insofar as $y \in G_0$, we may choose the subsequence j_k to ensure that there exists

$$\omega_{j_k} \in \Gamma_{j_k} \text{ with } |y - \omega_{j_k}| < \epsilon_k/k. \tag{4.31}$$

Now, by assumption, we can find admissible pairs (S_{1,t_k}, S_{2,t_k}) (for the sequence $\{\Omega_{j_k}\}_k$) centered at x with radius t_k satisfying

$$\liminf_{k\to\infty} \operatorname{dist}(y, (S_{1,t_k}, S_{2,t_k})_{\text{ep}}) \geq \ell_0. \tag{4.32}$$

Thus, by taking a further subsequence, relabelled again by $\{j_k\}_k$ (which preserves all the properties in (4.30)) and (4.31), we find $I_{t_k}^\pm \subset \Omega_{j_k}^\pm \cap \partial B(z_{j_k}, \widetilde{t}_k)$ satisfying

$$|x - z_{j_k}| \leq \epsilon_k, \quad |t_k - \widetilde{t}_k| \leq \epsilon_k, \text{ and } |\mathcal{H}^1(I_{t_k}^\pm) - \pi t_k| \leq \epsilon_k.$$

For k big enough and ϵ_k small enough, the end-points of $I_{s_k}^\pm$ and $I_{t_k}^\pm$ are far from y (by (4.29) and (4.32), respectively). Thus, we can assume that, say, any end-point z_k of these intervals satisfies $|y - z_k| \geq 0.9\ell_0$.

[4] To be clear, (S_1, S_2) is an admissible pair for the given sequence $\{\Omega_{j_k}^+\}_k$ of domains means that we can find a further subsequence $\{j_\ell\}_\ell$ of $\{j_k\}_k$ such that there exists circular arcs $I_{j_\ell}^\pm \subset \partial B(x_{j_\ell}, r_{j_\ell}) \cap \Omega_{j_\ell}^\pm$, with $x_{j_\ell} \in \Gamma_{j_\ell}$, such that $I_{j_\ell}^+$, $I_{j_\ell}^-$ converge to S_1, S_2 in Hausdorff distance, respectively.

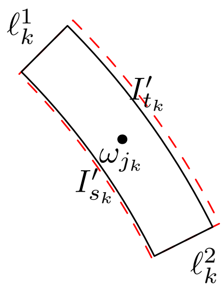

FIGURE 5. The figure depicts the solid black lined tubular neighbourhood $\widetilde{U}_k$, and the red dashed neighbourhood U_k.

Assuming $\epsilon_k \ll 1/k$, the arcs $I_{s_k}^{\pm}$ are essentially some perturbation of some arcs contained in $\partial B(x, s_k)$, while the arcs $I_{t_k}^{\pm}$ are also another small perturbation of other arcs from $\partial B(x, t_k)$. In fact, there is a thin tubular neighborhood U_k containing y that satisfies the following:

- $U_k = A(x, s_k + 2\epsilon_k, t_k - 2\epsilon_k) \cap V$, where V is the sector of $B(x, 2r)$ with axis equal to the line passing through x and y and such that its angle of aperture is $\ell_0/4r$, say.
- Associated with the arc $J_{s_k} := \partial B(x, s_k + 2\epsilon_k) \cap V \subset \partial U_k$ there is a close arc I'_{s_k} contained either in $I_{s_k}^+$ or $I_{s_k}^-$ such that $\mathrm{dist}_H(J_{s_k}, I'_{s_k}) \le c\epsilon_k r$.
- Associated with the arc $J_{t_k} := \partial B(x, t_k - 2\epsilon_k) \cap V \subset \partial U_k$ there is a close arc I'_{t_k} contained either in $I_{t_k}^+$ or $I_{t_k}^-$ such that $\mathrm{dist}_H(J_{t_k}, I'_{t_k}) \le c\epsilon_k r$.

Now we consider the tubular neighborhood $\widetilde{U}_k$ whose boundary is formed by the arcs I'_{s_k}, I'_{t_k} and two small segments ℓ_k^1, ℓ_k^2 that join the closest respective end-points of I'_{s_k} and I'_{t_k}, so that $\mathrm{dist}_H(U_k, \widetilde{U}_k) \lesssim \epsilon_k r$. See Fig. 5.

We distinguish two cases:

(1) In the first case, $I'_{s_k} \subset I_{s_k}^+$ and $I'_{t_k} \subset I_{t_k}^-$, or alternatively $I'_{s_k} \subset I_{s_k}^-$ and $I'_{t_k} \subset I_{t_k}^+$. In both situations, by connectivity there is a curve $\gamma_{j_k} \subset \partial\Omega_{j_k}^+ \cap \widetilde{U}_k$ that joins ℓ_k^1 with ℓ_k^2.

(2) In the second case, $I'_{s_k} \subset I_{s_k}^+$ and $I'_{t_k} \subset I_{t_k}^+$, or alternatively $I'_{s_k} \subset I_{s_k}^-$ and $I'_{t_k} \subset I_{t_k}^-$. By construction, the point $\omega_{j_k} \in \Gamma_{j_k}$ satisfies $|\omega_{j_k} - y| < \epsilon_k/k$, so we have $\omega_{j_k} \in \widetilde{U}_k$ and the distance of ω_{j_k} to any of the small segments ℓ_k^1, ℓ_k^2 from $\partial\widetilde{U}_k$ is at least $\ell_0/2$, say. Then, by the connectivity of $\partial\Omega_{j_k}^+$, there is a curve $\gamma_{j_k} \subset \partial\Omega_{j_k}^+ \cap \widetilde{U}_k$ that joints ω_{j_k} either to ℓ_k^1 or ℓ_k^2. So its diameter is at least $\ell_0/5$.

It is easy to check that the sequence of curves γ_{j_k} satisfy the properties asserted in the lemma. □

LEMMA 4.30. *Under the assumptions and notation of Lemma 4.28, let $x \in L \cap \operatorname{supp}\mu_0 \cap B_0$ and let $y \in \partial B(x, r) \cap G_0$ for some $r \in (0, 2r(B_0))$. Set*

$$\ell_0 = \inf_{(S_1, S_2)} \mathrm{dist}(y, (S_1, S_2)_{ep}),$$

where the infimum is taken over all admissible pairs centered at x with radius r.

The infimum is attained by an admissible pair centered at x with radius r, and if $\ell_0 > 0$ then there exists a subsequence of arcs $\gamma_{j_k} \subset \Gamma_{j_k}$ which converge in Hausdorff distance to an arc $I \subset \partial B(x, r)$ such that y is one of its end-points and $\mathcal{H}^1(I) \ge \ell_0/5$.

PROOF. The fact that the infimum is attained by an admissible pair (S_1, S_2) centered at x with radius $r > 0$ is an immediate consequence of the closedness of admissible pairs (and Lemma 4.13). Now suppose $\ell_0 = \mathrm{dist}(y, (S_1, S_2)_{\mathrm{ep}}) > 0$.

Consider the sequence of radii $s_k = r(1 - \frac{1}{k})$ and $t_k = r(1 + \frac{1}{k})$. Let (S_{1,s_k}, S_{2,s_k}) and (S_{1,t_k}, S_{2,t_k}) be sequences of admissible pairs centered at x with radii s_k and t_k respectively. By taking a subsequence, we may assume that these admissible pairs converge in Hausdorff metric to admissible pairs with centre x and radius r, say (S_1^-, S_2^-) and (S_1^+, S_2^+). By the minimal property of (S_1, S_2) we must have that $\operatorname{dist}(y, (S_1^\pm, S_2^\pm)_{\mathrm{ep}}) \geq \operatorname{dist}(y, (S_1, S_2)_{\mathrm{ep}})$. By the closeness property of the set of admissible pairs, we infer that

$$\liminf_{k\to\infty} \operatorname{dist}(y, (S_{1,s_k}, S_{2,s_k})_{\mathrm{ep}}) \geq \operatorname{dist}(y, (S_1, S_2)_{\mathrm{ep}})$$

and

$$\liminf_{k\to\infty} \operatorname{dist}(y, (S_{1,t_k}, S_{2,t_k})_{\mathrm{ep}}) \geq \operatorname{dist}(y, (S_1, S_2)_{\mathrm{ep}}).$$

Consequently, we may apply Lemma 4.29 with $\ell_0 = \operatorname{dist}(y, (S_1, S_2)_{\mathrm{ep}})$. □

LEMMA 4.31. *Under the assumptions and notation of Lemma 4.28, let $x \in L \cap \operatorname{supp} \mu_0 \cap B_0$ and let $r \in (0, 2r(B_0))$. Then there exists an admissible pair of semicircumferences centered at x with radius r whose end-points belong to L.*

PROOF. Let $x \in L \cap \operatorname{supp} \mu_0 \cap B_0$ and let $r \in (0, 2r(B_0))$. Suppose that there does not exist an admissible pair of semicircumferences centered at x with radius r whose end-points belong to L. Let $y \in L \cap \partial B(x, r)$, then, by Lemma 4.30,

$$\ell_0 = \inf_{(S_1,S_2)} \operatorname{dist}(y, (S_1, S_2)_{\mathrm{ep}}) > 0$$

where the infimum is taken over all admissible pairs centered at x with radius r. Consequently, Lemma 4.30 ensures that there exists a subsequence of arcs $\gamma_{j_k} \subset \Gamma_{j_k}$ which converge in Hausdorff distance to an arc $I \subset \partial B(x, r)$ such that y is one of its end-points and I has length at least $\ell_0/5$.

By the closeness property of the admissibility property of pairs, for any small $\delta \in (0, r/2)$ there exists another radius $r_\delta \in (r - \delta, r)$ close enough to r such that, denoting by y_δ the point in $L \cap \partial B(x, r_\delta)$ which is closest to y, any admissible pair (S_1^δ, S_2^δ) of semicircumferences contained in $\partial B(x, r_\delta)$ satisfies $\operatorname{dist}(y_\delta, (S_1^\delta, S_2^\delta)_{\mathrm{ep}}) \geq \ell_0/2$. Observe that $y_\delta \in G_0$ and then, by applying Lemma 4.30 to the subsequence of domains $\Omega_{j_k}^+$ (observe that the sets G^+, G^-, G_0 associated with the subsequence are the same as the ones associated with the original sequence $\{\Omega_j\}_j$), we infer that there is a subsequence of arcs $\gamma_{\delta, j_k'} \subset \Gamma_{j_k'}$ which converge in Hausdorff distance to an arc $I_\delta \subset \partial B(x, r_\delta)$ such that y_δ is one of its end-points and I_δ has length at least $\ell_0/10$. By renaming the subsequence, we can assume that $\{j_k'\}_k$ coincides with $\{j_k\}_k$.

By iterating the preceding argument, we still find another $r^\delta \in (r, r + \delta)$ close enough to r for which, after renaming the subsequence and denoting by y^δ the point in $L \cap \partial B(x, r^\delta)$ which is closest to y, there is a family arcs $\gamma_{j_k}^\delta \subset \Gamma_{j_k}$ which converge in Hausdorff distance to an arc $I^\delta \subset \partial B(x, r^\delta)$ such that y^δ is one of its end-points and has length at least $\ell_0/10$.

Let $x' \in L \cap \operatorname{supp} \mu_0$ with $x' \neq x$. Suppose that x' and y are in the same half-line contained in L with end-point equal to x (i.e., x' and y are at the same side of x in L). Otherwise, in the arguments above we interchange y with the other point from $\partial B(x, r) \cap L$. It is easy to check that any circumference $\partial B(x'', r')$, with δ small enough and x'' close enough to x', intersects at least two of the arcs I, I_δ, I^δ for all r' in some interval H of width bounded from below depending on the relative position of $x, x', y, y_\delta, y^\delta$. In fact, the same phenomenon happens replacing the arcs I, I_δ, I^δ by the curves $\gamma_{j_k}, \gamma_{\delta, j_k}, \gamma_{j_k}^\delta$, assuming k big enough. From this fact, one deduces easily that there exists some $r' \in H$ such that there is no admissible pair of semicircumferences with center x' and radius r' (associated to the sequence of domains $\{\Omega_{j_k}\}_k$), which is in contradiction with the hypothesis (2) in Lemma 4.28. □

LEMMA 4.32. *Under the assumptions of Lemma 4.28, let $x \in L \cap \operatorname{supp} \mu_0 \cap B_0$ and let $r \in (0, 2r(B_0))$. Assume that L coincides with the horizontal axis and suppose that $\partial B(x, r) \cap G_0 \cap \mathbb{R}^2_+ \neq \varnothing$ (recall that we assume $\mathbb{R}^2_+$ to be open) and let $y \in \partial B(x, r) \cap G_0 \cap \mathbb{R}^2_+$. Then there exists a sequence of arcs $\gamma_{j_k} \subset \Gamma_{j_k}$ which converge in Hausdorff distance to an arc $I \subset \partial B(x, r)$ such that y is one of its end-points and has length at least $\operatorname{dist}(y, L)/5$.*

PROOF. Fix a subsequence Ω_{j_k} of the domains Ω_j. This subsequence Ω_{j_k} again satisfies the assumptions of Lemma 4.28 (with the same choices of sets G^+, G^- and G_0). By Lemma 4.31, for every $\delta \in (0, r/2)$, there are admissible pairs $(S_{1,\delta}, S_{2,\delta})$ for Ω_{j_k} centered at x with respective radii equal to any number $s \in (r - \delta, r + \delta)$ such that their end-points belong all to L. Consequently, for sufficiently large k we may certainly find sequences of admissible pairs $(S_{1,r+1/k}, S_{2,r+1/k})$ and $(S_{1,r-1/k}, S_{2,r-1/k})$ centred at x with radii $r+1/k$ and $r-1/k$ respectively, and with end points on H (and so at a distance $\operatorname{dist}(y,L)$ from y). Thus, we may apply Lemma 4.29 with $\ell_0 = \operatorname{dist}(y,L)$, which completes the proof. $\square$

We are now in a position to complete the proof of Lemma 4.28, which is an immediate consequence of the following statement.

Lemma 4.33. *Under the assumptions of Lemma 4.28, let $x \in L \cap \operatorname{supp}\mu_0 \cap B_0$ and let $r \in (0, 2r(B_0))$. Then $\partial B(x,r) \cap G_0 \cap \mathbb{R}^2_+ = \varnothing$ (assuming L to be the horizontal axis).*

Proof. Suppose that $y \in \partial B(x,r) \cap G_0 \cap \mathbb{R}^2_+$. By Lemma 4.32, there exists a sequence of arcs $\gamma_{j_k} \subset \Gamma_{j_k}$ which converge in Hausdorff distance to an arc $I \subset \partial B(x,r)$ such that y is one of its end-points and has length at least $\operatorname{dist}(y,L)/5$.

Let $x' \in \operatorname{supp}\mu_0 \cap L$, with $x' \neq x$, and let y' be the middle point of the arc I (we may assume that $y' \notin L$), and let $r' = |x'-y'|$, so that $\partial B(x',r')$ intersects I in the middle point. By connectivity arguments, the existence of the curves γ_{j_k} given by Lemma 4.32 implies that, for the subsequence of domains Ω_{j_k}, there does not exist an admissible pair of semicircumferences centered at x' with radius r' whose end-points belong to L. This fact contradicts Lemma 4.31. $\square$

With Lemma 4.28 proved, we are now in a position to complete the proof of Lemma 4.22.

Proof of Lemma 4.22. By renormalizing it suffices to prove the lemma for the ball $B_0 := B(0,1)$. We argue by contradiction: we suppose that there exists an $\epsilon > 0$ such that for all $j \in \mathbb{N}$, there exists a Jordan domain Ω_j^+ with $\Gamma_j := \partial\Omega_j^+$ containing 0 supporting a measure μ_j with C_0-linear growth with $\mu_j(B_0) > c_0$, so that we have

$$\int_{7B_0} \int_0^7 \left(\epsilon_j(x,r)^2 + \alpha_j^+(x,r)^2\right) \frac{dr}{r}\, d\mu_j(x) \leq \frac{1}{j}\, \mu_j(7_0),$$

and

$$\beta_{\infty,\Gamma_j}(B_0) > \epsilon.$$

Here we denote by $\epsilon_j(x,r)$ and α_j^+ the coefficients $\epsilon(x,r)$ and α^+ associated with Ω_j^+.

We first apply Lemma 4.24 to pass to a subsequence of the domains such that, with $\Omega_j^- = \mathbb{R}^2 \backslash \overline{\Omega_j^+}$,

$$\Omega_j^{\pm} \to G^{\pm} \text{ and } \Gamma_j \to G_0$$

locally as $j \to \infty$. By passing to a further subsequence if necessary, we may assume that μ_j converge weakly to a measure μ_0 with C_0-linear growth satisfying $\mu(\overline{B_0}) \geq c_0$. Applying Lemma 4.25, we infer that the assumptions of Lemma 4.26 are satisfied with B_0 replaced by the ball $2B_0$, so there is a line $L \subset G_0$ with $\mu_0(L \cap 2B_0) > 0$.

Observe now that we can also apply Lemma 4.25 to any subsequence of the domains. In particular, from the conclusion (2) of Lemma 4.25 applied to a given subsequence, we infer that the assumption (2) of Lemma 4.28 also holds, again with the ball $2B_0$ playing the role of B_0 in Lemma 4.28. Therefore, applying Lemma 4.28 to the ball $2B_0$ that satisfies $\mu_0(L \cap 2B_0) > 0$, we have that $G_0 \cap 2B_0 \subset L$. Consequently, $\Gamma_j \cap \overline{B_0}$ converges in Hausdorff distance to a subset of L, which contradicts $\beta_{\infty,\Gamma_j}(B_0) > \epsilon$ for sufficiently large j. $\square$

Part II: From local flatness to rectifiability

9. The smooth square function on Lipschitz graphs

Recall that, given an integrable C^∞ function $\psi : \mathbb{R}^2 \to \mathbb{R}$, an open set $\Omega^+ \subset \mathbb{R}^2$ and $x \in \mathbb{R}^2$, $r > 0$, we denote

$$c_\psi = \int_{y\in\mathbb{R}^2_+} \psi(y)\, dy, \qquad \mathfrak{a}_\psi(x,r) = \left| c_\psi - \frac{1}{r^2} \int_{\Omega^+} \psi\left(\frac{y-x}{r}\right) dy \right|.$$

We also set

$$\mathcal{A}_\psi(x)^2 = \int_0^\infty \mathfrak{a}_\psi(x,r)^2 \frac{dr}{r}.$$

Let us remark that we do not require ψ to be radial.

We fix an even C^∞ function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{1}_{[-1,1]} \leq \varphi \leq \mathbb{1}_{[-1.1,1.1]}$, and we denote

$$\varphi_r(x) = \frac{1}{r}\varphi\Big(\frac{x}{r}\Big), \qquad \text{for } x \in \mathbb{R},\, r > 0.$$

Our objective in this section is to prove the following.

LEMMA 4.34. *Consider a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ with compact support and let $\Gamma \subset \mathbb{R}^2$ be its Lipschitz graph. Let $\Omega^+ = \{(x,y) \in \mathbb{R}^2 : y > f(x)\}$ and $\Omega^- = \{(x,y) \in \mathbb{R}^2 : y < f(x)\}$. Let φ be a function as above and let*

$$\psi(x) = \varphi(|x|), \qquad \text{for } x \in \mathbb{R}^2.$$

Let $\mathcal{A}_\psi$ and $\mathfrak{a}_\psi$ be the associated coefficients defined above. There exists some $\alpha_0 > 0$ such that if $\|\nabla f\|_\infty \leq \alpha_0$, then

$$\int_\Gamma \mathcal{A}_\psi(x)^2 \, d\mathcal{H}^1(x) \approx \|\nabla f\|^2_{L^2(\mathbb{R})}.$$

We will prove this result by using the Fourier transform. This will play an essential role in the proof of Main Lemma 4.7. We need first some auxiliary results.

LEMMA 4.35. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function with compact support. Then we have*

$$\int_\mathbb{R}\int_0^\infty \left|\frac{f * \varphi_r(x) - c(\varphi) f(x)}{r}\right|^2 \frac{dr}{r}\, dx = c\,\|\nabla f\|^2_{L^2(\mathbb{R})},$$

where $c(\varphi) = \int_\mathbb{R} \varphi \, dx$ and $c > 0$.

PROOF. By Plancherel, we have

$$\begin{aligned}\int_\mathbb{R}\int_0^\infty \left|\frac{f * \varphi_r(x) - c(\varphi) f(x)}{r}\right|^2 \frac{dr}{r}\, dx &= \int_\mathbb{R}\int_0^\infty \left|\frac{\widehat{f}(\xi)\widehat{\varphi}(r\xi) - \widehat{f}(\xi)\widehat{\varphi}(0)}{r}\right|^2 \frac{dr}{r}\, d\xi \\ &= \int_\mathbb{R} |\widehat{f}(\xi)|^2 \int_0^\infty \left|\widehat{\varphi}(r\xi) - \widehat{\varphi}(0)\right|^2 \frac{dr}{r^3}\, d\xi.\end{aligned}$$

By the change of variable $r|\xi| = t$, we get

$$\int_0^\infty \left|\widehat{\varphi}(r\xi) - \widehat{\varphi}(0)\right|^2 \frac{dr}{r^3} = |\xi|^2 \int_0^\infty \left|\widehat{\varphi}(t) - \widehat{\varphi}(0)\right|^2 \frac{dt}{t^3} =: \tilde{c}(\varphi)|\xi|^2,$$

where $0 < \tilde{c}(\varphi) < \infty$, since $\widehat{\varphi}(t) - \widehat{\varphi}(0) = O(t^2)$ as $t \to 0$ (because φ is an even function in the Schwartz class). Hence,

$$\int_\mathbb{R}\int_0^\infty \left|\frac{f * \varphi_r(x) - c(\varphi) f(x)}{r}\right|^2 \frac{dr}{r}\, dx = \tilde{c}(\varphi) \int_\mathbb{R} |\xi \widehat{f}(\xi)|^2 \, d\xi = c\|\nabla f\|^2_{L^2(\mathbb{R})}.$$

□

LEMMA 4.36. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function with compact support. Then we have*

(4.33)
$$\int_0^\infty\int_\mathbb{R}\int_{y\in\mathbb{R}:|y-x|\leq r} \left|\frac{c(\varphi)^{-1}(\varphi_r * f')(x)(y-x) + f(x) - f(y)}{r}\right|^2 \frac{dy}{r}\, dx\, \frac{dr}{r} = c\|\nabla f\|^2_{L^2(\mathbb{R})}.$$

PROOF. By replacing φ by $c(\varphi)^{-1}\varphi$ if necessary, we may assume that $\int \varphi\, dx = 1$. This is due to the fact that, as we shall see below, the assumption that $\mathbb{1}_{[-1,1]} \leq \varphi \leq \mathbb{1}_{[-1.1,1.1]}$ is not necessary for the validity of this lemma.

Appealing to the change of variable $z = y - x$ and Fubini's theorem, the left hand side of (4.33) (with $c(\varphi) = 1$) equals

$$\begin{aligned}&\int_0^\infty\int_{z\in\mathbb{R}:|z|\leq r}\int_{x\in\mathbb{R}} \left|\frac{(\varphi_r * f')(x) z + f(x) - f(x+z)}{r}\right|^2 dx\, \frac{dz}{r}\, \frac{dr}{r} \\ &\overset{\text{Plancharel}}{=} \int_0^\infty\int_{z\in\mathbb{R}:|z|\leq r}\int_{\xi\in\mathbb{R}} \left|\frac{2\pi i \xi z\, \widehat{\varphi}(r\xi)\, \widehat{f}(\xi) + \widehat{f}(\xi) - e^{2\pi i \xi z}\widehat{f}(\xi)}{r}\right|^2 \frac{dz}{r}\, d\xi\, \frac{dr}{r}.\end{aligned}$$

Using Fubini's theorem to interchange the inner two integrals, and the changes of variable $w = \xi z$, $s = |\xi| r$, we infer from the fact that φ (and so $\widehat{\varphi}$) is even that the last triple integral equals

$$\int_{\xi\in\mathbb{R}}\int_0^\infty\int_{w\in\mathbb{R}:|w|\le s}\left|2\pi iw\,\widehat{\varphi}(s)\,\widehat{f}(\xi)+\widehat{f}(\xi)-e^{2\pi iw}\widehat{f}(\xi)\right|^2\,|\xi|^2\,\frac{ds}{s^4}\,dw\,d\xi$$
$$=\int_{\xi\in\mathbb{R}}|\xi\,\widehat{f}(\xi)|^2\,d\xi\,\int_0^\infty\int_{w\in\mathbb{R}:|w|\le s}\left|2\pi iw\,\widehat{\varphi}(s)+1-e^{2\pi iw}\right|^2\,dw\,\frac{ds}{s^4}.$$

Hence, to prove the lemma it suffices to show that the last double integral

$$I:=\int_0^\infty\int_{w\in\mathbb{R}:|w|\le s}\left|2\pi iw\,\widehat{\varphi}(s)+1-e^{2\pi iw}\right|^2\,dw\,\frac{ds}{s^4}$$

is absolutely convergent and positive. That this is positive is immediate. To show that this is absolutely convergent, we split it as follows:

$$I=\int_0^\infty\int_{|w|\le\min(s,1)}\cdots+\int_1^\infty\int_{1\le|w|\le s}\cdots=:I_1+I_2.$$

First we estimate I_2:

$$I_2\lesssim\int_1^\infty\int_{1\le|w|\le s}\left(1+\left|w\,\widehat{\varphi}(s)\right|^2\right)dw\,\frac{ds}{s^4}$$
$$\lesssim\int_1^\infty\frac{ds}{s^3}+\int_1^\infty\int_{1\le|w|\le s}\left|s\,\widehat{\varphi}(s)\right|^2dw\,\frac{ds}{s^4}\lesssim 1+\int_1^\infty\left|\widehat{\varphi}(s)\right|^2\frac{ds}{s}\lesssim 1$$

Concerning I_1, we have

$$I_1\lesssim\int_0^\infty\int_{|w|\le\min(s,1)}\left|2\pi iw+1-e^{2\pi iw}\right|^2\,dw\,\frac{ds}{s^4}$$
$$+\int_0^\infty\int_{|w|\le\min(s,1)}|2\pi iw\,(\widehat{\varphi}(s)-1)|^2\,dw\,\frac{ds}{s^4}. \tag{4.34}$$

The first term on the right hand side satisfies

$$\int_0^\infty\int_{|w|\le\min(s,1)}\left|2\pi iw+1-e^{2\pi iw}\right|^2\,dw\,\frac{ds}{s^4}\le\int_{|w|\le1}\int_{s\ge|w|}\left|2\pi iw+1-e^{2\pi iw}\right|^2\,\frac{ds}{s^4}\,dw$$
$$\lesssim\int_{|w|\le1}\left|2\pi iw+1-e^{2\pi iw}\right|^2\,\frac{dw}{|w|^3}\lesssim 1,$$

taking into account that $2\pi iw+1-e^{2\pi iw}=O(w^2)$ as $w\to0$. Finally we turn our attention to the second term on the right hand side of (4.34):

$$\int_0^\infty\int_{|w|\le\min(s,1)}|2\pi iw\,(\widehat{\varphi}(s)-1)|^2\,dw\,\frac{ds}{s^4}\lesssim\int_{|w|\le1}|w|^2\,dw\int_0^\infty|\widehat{\varphi}(s)-1|^2\,\frac{ds}{s^4}$$
$$\lesssim\int_0^\infty|\widehat{\varphi}(s)-1|^2\,\frac{ds}{s^4}.$$

Since $\varphi\in C^\infty$ is even, and $\widehat{\varphi}(0)=1$, we have $\widehat{\varphi}(s)-1=O(s^2)$ as $s\to0$, and so the last integral is finite. So $I_2<\infty$ and the proof of the lemma is concluded. $\square$

Lemma 4.37. *Let $f:\mathbb{R}\to\mathbb{R}$ be a Lipschitz function with compact support with $\|\nabla f\|_\infty\le 1/10$. For $x=(x_1,x_2)\in\mathbb{R}^2$, denote*

$$\rho(x)=\varphi(x_1)\,\varphi(x_2).$$

Then we have

$$\mathfrak{a}_\rho(x,r)=\frac{\varphi_r*f(x_1)-c(\varphi)\,f(x_1)}{r}\quad\text{for all } x\in\mathbb{R}^2 \text{ in the graph of } f \text{ and all } r>0$$

and

$$\int_{\mathbb{R}}\mathcal{A}_\rho((x_1,f(x_1)))^2\,dx_1=\int_{\mathbb{R}}\int_0^\infty\left|\frac{f*\varphi_r(x_1)-c(\varphi)f(x_1)}{r}\right|^2\,\frac{dr}{r}\,dx_1,$$

where $c(\varphi)=\int_{\mathbb{R}}\varphi\,dx$.

PROOF. Observe that

$$\begin{aligned}
c_\rho - \frac{1}{r^2}\int_{\Omega^+} \rho\Big(\frac{y-x}{r}\Big)\,dy &= \frac{1}{r^2}\int_{y_1\in\mathbb{R}}\int_{y_2>f(x_1)} \varphi\Big(\frac{y_1-x_1}{r}\Big)\varphi\Big(\frac{y_2-x_2}{r}\Big)\,dy_2\,dy_1\\
&\quad - \frac{1}{r^2}\int_{y_1\in\mathbb{R}}\int_{y_2>f(y_1)} \varphi\Big(\frac{y_1-x_1}{r}\Big)\varphi\Big(\frac{y_2-x_2}{r}\Big)\,dy_2dy_1\\
&= \int_{y_1\in\mathbb{R}} \varphi_r(y_1-x_1)\int_{f(x_1)}^{f(y_1)} \varphi_r(y_2-x_2)\,dy_2dy_1
\end{aligned}$$

Observe also that, if $\varphi_r(y_1-x_1)\neq 0$, then because $\|\nabla f\|_\infty \le 1/10$,

$$\varphi_r(y_2-x_2) = \frac{1}{r} \quad \text{for } y_2\in[f(x_1),f(y_1)].$$

As a consequence

$$c_\rho - \frac{1}{r^2}\int_{\Omega^+} \rho\Big(\frac{y-x}{r}\Big)\,dy = \int_{y_1\in\mathbb{R}} \varphi_r(y_1-x_1)\frac{f(y_1)-f(x_1)}{r}dy_1 = \frac{\varphi_r * f(x_1) - c(\varphi)\,f(x_1)}{r}.$$

Hence,

$$\mathcal{A}_\varphi(x)^2 = \int_0^\infty \mathfrak{a}_\varphi(x,r)^2\,\frac{dr}{r} = \int_0^\infty \left|\frac{\varphi_r * f(x_1) - c(\varphi)\,f(x_1)}{r}\right|^2\frac{dr}{r}.$$

Integrating with respect to x_1 in $\mathbb{R}$, the lemma follows. □

LEMMA 4.38. *Let $f:\mathbb{R}\to\mathbb{R}$ be a Lipschitz function with compact support with $\|\nabla f\|_\infty \le 1/10$ and let $\Gamma\subset\mathbb{R}^2$ be its Lipschitz graph. For $x=(x_1,x_2)\in\mathbb{R}^2$, denote*

$$\rho(x) = \varphi(x_1)\,\varphi(x_2) \quad \textit{and} \quad \psi(x)=\varphi(|x|).$$

Then we have

$$\int_\Gamma |\mathcal{A}_\rho(x) - \mathcal{A}_\psi(x)|^2\,d\mathcal{H}^1(x) \lesssim \|\nabla f\|_\infty^4\,\|\nabla f\|_{L^2(\mathbb{R})}^2.$$

PROOF. For $r>0$, $x\in\mathbb{R}^2$, we denote

$$\rho_r(x) = \frac{1}{r^2}\rho\Big(\frac{x}{r}\Big), \qquad \psi_r(x) = \frac{1}{r^2}\psi\Big(\frac{x}{r}\Big), \qquad \varphi_r(x_1) = \frac{1}{r}\varphi\Big(\frac{x_1}{r}\Big).$$

Then we have

$$\big|\mathfrak{a}_\rho(x,r) - \mathfrak{a}_\psi(x,r)\big| \le \big|(\rho_r * \mathbb{1}_{\Omega^+} - c_\rho) - (\psi_r * \mathbb{1}_{\Omega^+} - c_\psi)\big|. \tag{4.35}$$

For $x\in\Gamma$, $r>0$, we denote by $L_{x,r}$ the line passing through x with slope equal to $c(\varphi)^{-1}(\varphi_r * f')(x_1)$, and we let $H^+_{x,r}$, $H^-_{x,r}$ be two complementary half planes whose common boundary is $L_{x,r}$, so that $H^+_{x,r}$ is above $L_{x,r}$ and $H^-_{x,r}$ is below $L_{x,r}$.

Observe that, by the radial symmetry of ψ,

$$c_\psi = \int_{y\in\mathbb{R}^2_+} \psi(y)\,dy = \psi_r * \mathbb{1}_{H_{x,r}}(x) \quad \text{for all } x\in\mathbb{R}^2 \text{ and } r>0.$$

We claim that the same identity holds replacing ψ by ρ. To check this, suppose that $x=0$ for easiness of notation and let $y_2 = b\,y_1$ be equation of the line $L_{0,r}$. Then, by the evenness of φ we have

$$\begin{aligned}
\rho_r * \mathbb{1}_{H_{0,r}}(0) &= \int \varphi_r(y_1)\int_{y_2>by_1} \varphi_r(y_2)\,dy_2\,dy_1\\
&= \frac{1}{2}\int \varphi_r(y_1)\int_{y_2>by_1}\varphi_r(y_2)\,dy_2\,dy_1 + \frac{1}{2}\int \varphi_r(y_1)\int_{y_2>-by_1}\varphi_r(y_2)\,dy_2\,dy_1\\
&= \frac{1}{2}\int \varphi_r(y_1)\int_{y_2>by_1}\varphi_r(y_2)\,dy_2\,dy_1 + \frac{1}{2}\int \varphi_r(y_1)\int_{y_2<by_1}\varphi_r(y_2)\,dy_2\,dy_1\\
&= \frac{1}{2}\int \rho_r(y)\,dy = \int_{\mathbb{R}^2_+}\rho_r(y)\,dy = c_\rho,
\end{aligned}$$

which proves the claim.

From the above identities and (4.35), for $x \in \Gamma$, we obtain

$$\begin{aligned}\tag{4.36} \big|\mathfrak{a}_\rho(x,r) - \mathfrak{a}_\psi(x,r)\big| &\le \big|\rho_r * (\mathbb{1}_{\Omega^+} - \mathbb{1}_{H^+_{x,r}})(x) - \psi_r * (\mathbb{1}_{\Omega^+} - \mathbb{1}_{H^+_{x,r}})(x)\big| \\ &= \big|(\rho_r - \psi_r) * (\mathbb{1}_{\Omega^+} - \mathbb{1}_{H^+_{x,r}})(x)\big| \\ &\le \int_{\Omega^+ \Delta H^+_{x,r}} |\rho_r(y-x) - \psi_r(y-x)|\, dy.\end{aligned}$$

But now observe that, if $|x-y| < 3r$, then using the fact that $\|\nabla f\|_\infty < 1/10$, we have $\varphi_r(y_2 - x_2) = \frac{1}{r}$ for all $x \in \Gamma$ and $y \in \Omega^+\Delta H^+_{x,r}$. Thus, by the definition of ρ and ψ,

$$\rho_r(y-x) - \psi_r(y-x) = \varphi_r(y_1-x_1)\varphi_r(y_2-x_2) - \frac{1}{r}\varphi_r(|y-x|) = \frac{1}{r}\left(\varphi_r(y_1-x_1) - \varphi_r(|y-x|)\right).$$

Still for $x \in \Gamma$ and $y \in \Omega^+\Delta H^+_{x,r}$, notice that if $|x-y| \le r/2$, then $\varphi_r(y_1-x_1) = \varphi_r(|y-x|) = \frac{1}{r}$ and thus $\rho_r(y-x) - \psi_r(y-x) = 0$; while if $|x-y| \ge r/2$

$$\big|\rho_r(y-x) - \psi_r(y-x)\big| \le \frac{1}{r}\|\nabla\varphi_r\|_\infty \big|(y_1-x_1) - |y-x|\big| \lesssim \frac{|y_2-x_2|^2}{r^4}.$$

Since $\operatorname{supp}\rho_r \cup \operatorname{supp}\psi_r \subset B(0,3r)$, in any case we get

$$\big|\rho_r(y-x) - \psi_r(y-x)\big| \lesssim \frac{(\|\nabla f\|_\infty\, r)^2}{r^4} = \frac{\|\nabla f\|_\infty^2}{r^2} \quad \text{for } x \in \Gamma \text{ and } y \in \Omega^+\Delta H^+_{x,r}.$$

Plugging this estimate into (4.36), we obtain

$$\big|\mathfrak{a}_\rho(x,r) - \mathfrak{a}_\psi(x,r)\big| \lesssim \frac{\|\nabla f\|_\infty^2}{r^2}\, \mathcal{H}^2\big((\Omega^+\Delta H^+_{x,r}) \cap B(x,3r)\big).$$

Next, using the fact that the equation of the line $L_{x,r}$ is

$$y_2 = c(\varphi)^{-1}(\varphi_r * f')(x_1)(y_1-x_1) + f(x_1),$$

we get

$$\begin{aligned}\mathcal{H}^2\big((\Omega^+\Delta H^+_{x,r}) \cap B(x,3r)\big) &\le \int_{|y_1-x_1|\le 3r} |c(\varphi)^{-1}(\varphi_r * f')(x_1)(y_1-x_1) + f(x_1) - f(y_1)|\, dy_1 \\ &\lesssim r^{1/2}\left(\int_{|y_1-x_1|\le 3r} |c(\varphi)^{-1}(\varphi_r * f')(x_1)(y_1-x_1) + f(x_1) - f(y_1)|^2\, dy_1\right)^{1/2}.\end{aligned}$$

Hence,

$$\begin{aligned}&\big|\mathfrak{a}_\rho(x,r) - \mathfrak{a}_\psi(x,r)\big| \\ &\qquad\lesssim \frac{\|\nabla f\|_\infty^2}{r^{3/2}}\left(\int_{|y_1-x_1|\le 3r} |c(\varphi)^{-1}(\varphi_r * f')(x_1)(y_1-x_1) + f(x_1) - f(y_1)|^2\, dy_1\right)^{1/2}.\end{aligned}$$

Therefore,

$$\begin{aligned}&|\mathcal{A}_\rho(x) - \mathcal{A}_\psi(x)| \\ &\quad = \left|\left(\int_0^\infty \mathfrak{a}_\rho(x,r)^2\,\frac{dr}{r}\right)^{1/2} - \left(\int_0^\infty \mathfrak{a}_\psi(x,r)^2\,\frac{dr}{r}\right)^{1/2}\right| \\ &\quad \le \left(\int_0^\infty |\mathfrak{a}_\rho(x,r) - \mathfrak{a}_\psi(x,r)|^2\,\frac{dr}{r}\right)^{1/2} \\ &\quad \lesssim \|\nabla f\|_\infty^2\left(\int_0^\infty\int_{|y_1-x_1|\le 3r} |c(\varphi)^{-1}(\varphi_r * f')(x_1)(y_1-x_1) + f(x_1) - f(y_1)|^2\, dy_1\,\frac{dr}{r^3}\right)^{1/2}.\end{aligned}$$

Squaring and integrating on x and applying Lemma 4.36, we get

$$\begin{aligned}&\int_\Gamma |\mathcal{A}_\rho(x) - \mathcal{A}_\psi(x)|^2\, d\mathcal{H}^1(x) \approx \int |\mathcal{A}_\rho(x) - \mathcal{A}_\psi(x)|^2\, dx_1 \\ &\quad\lesssim \|\nabla f\|_\infty^4 \int_{\mathbb{R}}\int_0^\infty\int_{|y_1-x_1|\le 3r} |c(\varphi)^{-1}(\varphi_r * f')(x_1)(y_1-x_1) + f(x_1) - f(y_1)|^2\, dy_1\,\frac{dr}{r^3}\, dx_1 \\ &\quad\approx \|\nabla f\|_\infty^4\, \|\nabla f\|_{L^2(\mathbb{R})}^2.\end{aligned}$$

$\square$

Proof of Lemma 4.34. By Lemma 4.35 and Lemma 4.37, we have

$$\int_\Gamma \mathcal{A}_\rho(x)^2 \, d\mathcal{H}^1(x) \approx \|\nabla f\|_{L^2(\mathbb{R})}^2.$$

On the other hand, by Lemma 4.38,

$$\int_\Gamma |\mathcal{A}_\rho(x) - \mathcal{A}_\psi(x)|^2 \, d\mathcal{H}^1(x) \lesssim \|\nabla f\|_\infty^4 \, \|\nabla f\|_{L^2(\mathbb{R})}^2.$$

Hence,

$$\int_\Gamma \mathcal{A}_\psi(x)^2 \, d\mathcal{H}^1(x) \leq 2 \int_\Gamma \mathcal{A}_\rho(x)^2 \, d\mathcal{H}^1(x) + 2 \int_\Gamma |\mathcal{A}_\rho(x) - \mathcal{A}_\psi(x)|^2 \, d\mathcal{H}^1(x) \lesssim \|\nabla f\|_{L^2(\mathbb{R})}^2.$$

In the converse direction, we have

$$\begin{aligned}\int_\Gamma \mathcal{A}_\psi(x)^2 \, d\mathcal{H}^1(x) &\geq \frac{1}{2} \int_\Gamma \mathcal{A}_\rho(x)^2 \, d\mathcal{H}^1(x) - \int_\Gamma \mathcal{A}_\psi(x)^2 \, d\mathcal{H}^1(x) \\ &\geq c_1 \|\nabla f\|_{L^2(\mathbb{R})}^2 - C \, \|\nabla f\|_\infty^4 \, \|\nabla f\|_{L^2(\mathbb{R})}^2.\end{aligned}$$

So if $\|\nabla f\|_\infty^4 \leq c_1/2C$, the lemma follows. $\square$

10. Beginning of the proof of Main Lemma 4.2

This and the remaining sections of the chapter are devoted to the proof of the Main Lemma 4.7. To this end, we need to construct a Lipschitz graph which locally covers a fairly big piece of the measure μ. We will achieve this through a construction stemming from works of David and Semmes in [**DS91**] and of Léger in [**Lé**].

10.1. Stopping cubes, trees and their properties. From now on, we assume that we are under the assumptions of Main Lemma 4.7. To prove this, we will perform a standard stopping time argument. Let us start by listing the various parameter which will be used.

(1) $\theta > 0$: this is the lower density parameter, which we will require to be small enough. At the very least, $\theta \leq c_0/100$, say. The parameter θ that appears in Main Lemma 4.7 is closely related, but we shall choose it to be small constant multiple of θ, so that in the proof that follows, whenever $\Theta_\mu(B) \gtrsim \theta$ (where the implicit constant is absolute), then $\beta_{\infty,\Gamma}(B) \leq \epsilon$.
(2) $\alpha > 0$: this is the parameter which regulates the slope of the Lipschitz graph that we will construct, will we always assume that $\alpha \ll 1$.
(3) $\epsilon > 0$: this is the parameter which regulates how small the $\beta_{\infty,\Gamma}$ coefficients and the square function $\mathcal{A}_\psi$ should be. The parameter ϵ will be chosen to depend on θ and α. At the very least, the reader should think that $\epsilon \ll \min(\theta, \alpha)$.

We consider the dyadic lattice associated to μ described in the Appendix, Theorem 5.3, and we will define a stopping-time region depending on the parameters above. By changing the starting generation k if necessary, we can assume that we have a maximal cube $Q_0 \in \mathcal{D}$ which coincides with $\operatorname{supp} \mu$. For simplicity, we will work with finite subfamilies of dyadic cubes, in the following sense. Let $k_1 \in \mathbb{Z}$, with $k_1 \gg k_0$. Set

$$\mathcal{D}^{k_1} := \bigcup_{k=k_0}^{k_1} \mathcal{D}_k.$$

Let L_0 be the minimising line of $\beta_{\infty,\Gamma}(Q_0)$. We define $\mathsf{Stop} = \mathsf{Stop}(k_1)$ to be the collection of maximal cubes Q in $\mathcal{D}^{k_1}$ which belongs to one of the following subfamilies.

(1) LD_0 ('low density'): these are the cubes $Q \in \mathcal{D}$ that satisfy

$$\Theta_\mu(2B_Q) < \theta.$$

(2) BA_0 ('big angles'): these are the cubes $Q \in \mathcal{D}$ which satisfy

$$\angle(L_Q, L_0) > \alpha.$$

(3) Out_0: these are the cubes $Q \in \mathcal{D}$ with
$$\ell(Q) < A_0^{-k_1}\ell(Q_0).$$

We then define $\mathsf{Tree} = \mathsf{Tree}(k_1)$ to be the subfamily of $\mathcal{D}$ so that no cube from Tree is properly contained in a cube from Stop. Note that $\mathsf{Stop} \subset \mathsf{Tree}$. We also denote
$$\mathsf{LD} = \mathsf{Stop} \cap \mathsf{LD}_0, \quad \mathsf{BA} = (\mathsf{Stop} \cap \mathsf{BA}_0) \setminus \mathsf{LD}, \quad \mathsf{Out} = (\mathsf{Stop} \cap \mathsf{Out}_0) \setminus (\mathsf{LD} \cup \mathsf{BA}).$$

It will often be convenient to identify LD with $\bigcup_{Q \in \mathsf{LD}} Q$, so $\mu(\mathsf{LD}) = \mu(\bigcup_{Q\in\mathsf{LD}} Q)$, and similarly with the sets BA and Out.

11. Construction of the Lipschitz graph

In this section we construct the Lipschitz graph. The Lipschitz graph will approximate well Q_0 at the scale of Stop; in particular it will depend on k_1. However, all the estimates will turn out to be independent from k_1.

By now the construction of the Lipschitz graph is quite standard, amd there are many presentations of it. We will follow Tolsa [To14], and will just state the relevant lemmas giving precise references where the reader can find the proof (with slightly different notation).

11.1. Preliminaries.

A first auxiliary result is the following well known fact from [DS91], Section 5.

Lemma 4.39. *Let L_1 and L_2 be lines in the plane. Suppose that x_1 and x_2 are two points in a subset X so that*

(1) $d_1 = \frac{|x_1 - x_2|}{\operatorname{diam}(X)} \in (0,1)$, and
(2) $\operatorname{dist}(x_i, L_j) < d_2 \operatorname{diam}(X)$ for $i = 1,2$ and $j = 1,2$, where $d_2 < d_1/4$.

Then for any $x \in L_2$,
$$\operatorname{dist}(x, L_1) \le d_2 \left(\frac{4}{d_1} \operatorname{dist}(x, X) + \operatorname{diam}(X) \right) \tag{4.37}$$

Lemma 4.40. *Let μ be a measure with 1-linear growth on $\mathbb{C}$ and let $\theta > 0$. There exists some constant $c_2 \ge 1$, depending only on c_0 and θ such that if B is a ball satisfying $\mu(B) > \theta\, r(B)$, then there are two balls B_1 and B_2 satisfying*

(a) $\operatorname{dist}(B_1, B_2) \ge 10 c_2^{-1} r(B)$.
(b) $\mu(B \cap B_j) \ge c_2^{-1} r(B)$ for $j = 1,2$.

For the proof, see for example Lemma 7.6 in [To14].

For a cube $Q \in \mathcal{D}$, we put
$$\beta_{\infty,\Gamma}(Q) := \beta_{\infty,\Gamma}(10B_Q). \tag{4.38}$$

We denote by L_Q a line that minimizes $\beta_{\infty,\Gamma}(Q)$. Notice that $\beta_{\infty,\Gamma}(Q) \lesssim \epsilon$ for every $Q \in \mathsf{Tree}$. Indeed, in the case that $Q \in \mathsf{Tree}\setminus\mathsf{Stop}$, this follows from the fact that $\Theta_\mu(10B_Q) \gtrsim \Theta_\mu(2B_Q) \ge \theta$ by the construction of Tree and taking also into account the assumptions in Main Lemma 4.7. In the case $Q \in \mathsf{Stop}$, the condition $\beta_{\infty,\Gamma}(Q) \lesssim \epsilon$ is derived from the analogous property for its parent, which does not belong to Stop (at the price of getting a somewhat worse implicit constant in the estimate).

The proof of the next lemma can be found as Lemma 7.14 in [To14].

Lemma 4.41. *Let $P, Q \in \mathsf{Tree} \setminus \mathsf{Stop}$ be so that $P \subset Q$. If $x \in L_P \cap B_P$, then*
$$\operatorname{dist}(x, L_Q) \le C(\theta)\, \epsilon\, \ell(Q).$$

We denote by Π (respectively $\Pi^\perp$) the standard orthogonal projection onto L_0 (respectively, $L_0^\perp$). For any given line L, we will denote by Π_L the orthogonal projection on L.

The next lemma can be found as Lemma 7.15 in [To14].

Lemma 4.42. *Let $Q, P \in \mathsf{Tree} \setminus \mathsf{Stop}$, $q \in L_Q \cap 2B_Q$ and $p \in L_P \cap 2B_P$. Then*
$$|\Pi^\perp(p) - \Pi^\perp(q)| \le C\alpha \left(|\Pi(p) - \Pi(q)| + 2\ell(P) + 2\ell(Q)\right).$$

11.2. A smoothing function. We define the following auxiliary function:

$$d(x) := \inf_{Q\in\mathsf{Tree}\setminus\mathsf{Stop}} (\operatorname{dist}(x,Q)+\ell(Q)) \text{ for } x\in\mathbb{C}. \tag{4.39}$$

Note that since $\mathsf{Tree} = \mathsf{Tree}(k_1)$ is finite, we have $d(x) > 0$ for all $x \in \mathbb{C}$.

The next lemma can be found as Lemma 7.19 of [To14].

LEMMA 4.43. *Let $\epsilon > 0$ and $\alpha > 0$ be sufficiently small. Then for any $x, y \in \mathbb{R}^2$,*

$$\left|\Pi^\perp(x) - \Pi^\perp(y)\right| \lesssim \alpha\,|\Pi(x)-\Pi(y)| + d(x) + d(y).$$

11.3. Whitney decomposition. Let $\mathcal{D}_{L_0}$ be the set of dyadic intervals in L_0. For $p \in L_0$, we denote

$$D(p) = \inf_{x\in\Pi^{-1}(p)} d(x),$$

and for $I \in \mathcal{D}_{L_0}$,

$$D(I) := \inf_{p\in I} D(x).$$

Set

$$\mathsf{W} := \{I \text{ maximal in } \mathcal{D}_{L_0} \,:\, \ell(I) < 20^{-1}\,D(I)\}.$$

We summarise the properties of the cubes in W in the following Lemma. We index W as $\{R_i\}_{i\in I_{\mathsf{W}}}$.

LEMMA 4.44. *The intervals R_i in W have disjoint interiors in L_0 and satisfy the following properties:*

(a) *If $x \in 15R_i$, then $5\ell(R_i) \le D(x) \le 50\ell(R_i)$.*

(b) *There exists an absolute constant $c > 1$ such that if $15R_i \cap 15R_j \ne \varnothing$, then*

$$c^{-1}\ell(R_i) \le \ell(R_j) \le c\,\ell(R_i). \tag{4.40}$$

(c) *For each $i \in I_{\mathsf{W}}$, there are at most N intervals R_j such that $15R_i \cap 15R_j \ne \varnothing$, where N is some absolute constant.*

The proof of this result is standard. See for example Lemma 7.20 in [To14].

Let

$$U_0 = L_0 \cap B_0, \text{ where } B_0 = B(\Pi(z_{Q_0}), 10\ell(Q_0)), \tag{4.41}$$

where z_{Q_0} is the center of Q_0.

We remark that, by adjusting the parameters of Main Lemma 4.7 if necessary, we can assume that B_0 coincides with the ball named B_0 in the statement of Main Lemma 4.7.

Certainly, $\operatorname{dist}(z_{Q_0}, L_0) < 10^{-1}\ell(Q_0)$, and so we have that

$$\Pi(Q_0) \subset \Pi(B_0) \subset 2B_0 \cap L_0. \tag{4.42}$$

Now set

$$I_0 := \{i \in I_{\mathsf{W}} \,:\, R_i \cap U_0 \ne \varnothing\}.$$

The next lemma is proven exactly as Lemma 7.21 from [To14].

LEMMA 4.45. *The following holds.*

- *If $i \in I_0$, then $\ell(R_i) \le \ell(Q_0)$ and $3R_i \subset L_0 \cap 1.2B_0$.*
- *If $i \notin I_0$, then*

$$\ell(R_i) \approx \operatorname{dist}(\Pi(z_{Q_0}), R_i) \approx |z_{Q_0} - x| \gtrsim \ell(Q_0) \textit{ for all } x \in R_i.$$

The next lemma can be found as Lemma 7.22 from [To14].

LEMMA 4.46. *Let $i \in I_0$; there exists a cube $Q = Q_i \in \mathsf{Tree}\setminus\mathsf{Stop}$ such that*

$$\ell(R_i) \lesssim \ell(Q_i) \lesssim \ell(R_i); \tag{4.43}$$

$$\operatorname{dist}(R_i, \Pi(Q_i)) \lesssim \ell(R_i). \tag{4.44}$$

For $i \in I_0$, denote by A_i the affine function $L_0 \to L_0^\perp$ whose graph is the line L_{Q_i}. Notice that, for each $i \in I_0$, A_i is Lipschitz with constant $\lesssim \alpha$.

On the other hand, for $i \in I_{\mathsf{W}} \setminus I_0$, we put $A_i \equiv 0$, so that its graph is just L_0.

The following lemma is proven in the same way as Lemma 7.23 in [To14].

Lemma 4.47. *If* $10R_i \cap 10R_j \neq \varnothing$ *for some* $i, j \in I_{\mathsf{W}}$, *then*

(1) $\operatorname{dist}(Q_i, Q_j) \lesssim \ell(R_i)$ *if, moreover,* $i, j \in I_0$;
(2) $|A_i(x) - A_j(x)| \leq c(\theta)\epsilon\ell(R_i)$ *for* $x \in 100R_i$;
(3) $|A_i'(x) - A_j'(x)| \leq c(\theta)\epsilon$.

11.4. Definition of A on L_0. In this subsection we complete the task of defining A on L_0. To do so, we recur to a standard construction involving a partition of unity adapted to the Whitney decomposition $\{R_i\}_{i\in I_{\mathsf{W}}}$; this construction goes as follows: for $i \in I_{\mathsf{W}}$, we find a function $\widetilde{\phi}_i \in C^\infty(L_0)$ such that

$$\mathbb{1}_{2R_i} \leq \widetilde{\phi} \leq \mathbb{1}_{3R_i} \tag{4.45}$$

$$\|\widetilde{\phi}_i'\|_\infty \lesssim \frac{1}{\ell(R_i)} \tag{4.46}$$

$$\|\widetilde{\phi}_i''\|_\infty \lesssim \frac{1}{\ell(R_i)^2}. \tag{4.47}$$

Then for each $i \in I_{\mathsf{W}}$, we put

$$\phi_i := \frac{\widetilde{\phi}_i}{\sum_{j\in I_{\mathsf{W}}} \widetilde{\phi}_j}. \tag{4.48}$$

Then it is immediate from the construction that $\{\phi_i\}_{i\in I_{\mathsf{W}}}$ is a partition of unity subordinated to $\{3R_i\}_{i\in I_{\mathsf{W}}}$. Moreover, properties (4.46) and (4.47) together with Lemma 4.44 give that

$$\|\nabla\phi_i\|_\infty \lesssim \ell(R_i)^{-1} \quad \text{and}$$
$$\|D^2\phi_i\|_\infty \lesssim \ell(R_i)^{-2}.$$

We now define a function $A_{k_1} = A : L_0 \to L_0^\perp$: if $x \in L_0$, we put

$$A(x) := \sum_{i\in I_{\mathsf{W}}} \phi_i(x) A_i(x) = \sum_{i\in I_0} \phi_i(x) A_i(x).$$

The following lemma, which follows in the same way as Lemmas 7.24 and 7.25 from [To14], will also be useful later on.

Lemma 4.48. *The function* $A : L_0 \to L_0^\perp$ *is supported on* $1.2B_0$ *and is Lipschitz with slope at most* $C\alpha$. *Moreover, if* $x \in 15R_i$ *for* $i \in I_{\mathsf{W}}$, *then*

$$|A''(x)| \lesssim \frac{\sqrt{\epsilon}}{\ell(R_i)}.$$

We will denote the graph of A by G_A, that is

$$G_A := \{(x, A(x)) \,|\, x \in L_0\}. \tag{4.49}$$

11.5. The Lipschitz graph G_A and $\operatorname{supp}\mu$ are close each other. The next four lemmas are proven almost exactly as Lemmas 7.28, 7.29, 7.30, and 7.31 in [To14].

Lemma 4.49. *Every any* $x \in 3B_{Q_0}$ *satisfies*

$$\operatorname{dist}(x, G_A) \lesssim d(x).$$

Lemma 4.50. *Let* $\epsilon > 0$ *be sufficiently small. If* $Q \in \mathsf{Tree} \setminus \mathsf{Stop}$ *and* $x \in G_A \cap 2B_Q$, *then*

$$\operatorname{dist}(x, L_Q) \lesssim \epsilon\,\ell(Q). \tag{4.50}$$

Lemma 4.51. *Let* $Q \in \mathsf{Tree}$. *Then every* $x \in Q$ *satisfies*

$$\operatorname{dist}(x, G_A) \lesssim \epsilon\ell(Q).$$

Lemma 4.52. *We have*

$$\operatorname{dist}(x, L_0) \leq c(\theta)\,\epsilon\,\ell(Q_0)$$

for all $x \in G_A$.

12. Small measure of LD and BA

12.1. LD has small measure. The following lemma follows the proof of Lemma 7.33 of [To14].

LEMMA 4.53. *Let $\epsilon > 0$ and $\theta > 0$ be sufficiently small. Then*

$$\sum_{Q \in \mathsf{LD}} \mu(Q) \lesssim \theta \mu(Q_0).$$

The implicit constant above is independent of k_1.

12.2. BA has small measure. Our main objective in this section is to prove the following.

LEMMA 4.54. *If ϵ is chosen sufficiently small, with respect to α and θ, then*

$$\sum_{Q \in \mathsf{BA}} \mu(Q) \lesssim \epsilon^{1/2} \mu(Q_0). \tag{4.51}$$

Note that the estimate in the lemma is independent of k_1.

Without loss of generality, we will assume that the line L_0 coincides with the horizontal axis of $\mathbb{R}^2$, and so $L_0^\perp$ is the vertical axis. We denote by $\mathfrak{a}_{\Gamma,\psi}$ and $\mathcal{A}_{\Gamma,\psi}$ the respective square functions $\mathfrak{a}_\psi$ and $\mathcal{A}_\psi$ associated with the open set $\Omega_\Gamma^+ \equiv \Omega^+$, whose boundary is Γ. The analogous square functions associated with the domain

$$\Omega_{G_A}^+ = \{x \in \mathbb{R}^2 : \Pi^\perp(x) > A(\Pi(x))\}$$

are denoted by $\mathfrak{a}_{G_A,\psi}$ and $\mathcal{A}_{G_A,\psi}$.

Recall that if B is a ball centered at a point $z_B \in \operatorname{supp} \mu$ and $\Theta_\mu(B) \gtrsim \theta$, then $\beta_{\infty,\Gamma}(B) \le \epsilon$. In particular, this implies that if L minimizes $\beta_{\infty,\Gamma}(B)$, then

$$B \setminus U_{2\epsilon r(B)}(L) \subset \Omega^+ \cup \Omega^-.$$

By connectivity, it is clear that each component of $B \setminus U_{2\epsilon r(B)}(L)$ is contained either in Ω^+ or in Ω^-. Also, assuming ϵ small enough, it is easy to check that the smallness of $\int_{r(B)/2}^{r(B)} \mathcal{A}_{\Gamma,\psi}(z_B, r)^2 \frac{dr}{r}$ implies that one of those components must be contained in Ω^+ and the other in Ω^-. In the particular case of B_0, by rotating the axes if necessary, we assume that

$$B_0 \cap \mathbb{R}^2_+ \setminus U_{2\epsilon r(B_0)}(L_0) \subset \Omega^+ \quad \text{and} \quad B_0 \cap \mathbb{R}^2_- \setminus U_{2\epsilon r(B_0)}(L_0) \subset \Omega^-. \tag{4.52}$$

To prove Lemma 4.54 we will show that if BA is big, then $\|A'\|_2^2$ is large, and that this in turn implies that a truncated version of $\int \mathcal{A}_{\Gamma,\psi}(x)\, d\mu(x)$ is also large, which cannot happen. The proof is split into several lemmas. The first one is the following; see Lemma 7.35 in [To14] for a proof.

LEMMA 4.55. *We have*

$$\sum_{Q \in \mathsf{BA}} \mu(Q) \le c\alpha^{-2} \|A'\|_2^2. \tag{4.53}$$

Recall that, by Lemma 4.34,

$$\int_{G_A} \mathcal{A}_{G_A,\psi}(x)^2 \, d\mathcal{H}^1(x) \approx \|A'\|_2^2, \tag{4.54}$$

where $\mathcal{A}_{G_A,\psi}$ stands for the square function $\mathcal{A}_\psi$ associated with the graph G_A. From this and the preceding lemma we infer that

$$\sum_{Q \in \mathsf{BA}} \mu(Q) \le \alpha^{-2} \int_{G_A} \mathcal{A}_{G_A,\psi}(x)^2 \, d\mathcal{H}^1(x). \tag{4.55}$$

Our next objective is to compare $\int_{G_A} \mathcal{A}_{G_A,\psi}(x)^2 \, d\mathcal{H}^1(x)$ with $\int \mathcal{A}_{\Gamma,\psi}(x)^2 \, d\mu(x)$.

We denote

$$\ell(x) := \frac{1}{50} D(x) = \frac{1}{50} D(\Pi(x)).$$

Observe that if $x \in 15I$, $I \in \mathsf{W}_{G_A}$, then

$$\frac{1}{10} \ell(I) \le \ell(x) \le \ell(I),$$

by Lemma 4.44. We set

$$\widetilde{\mathcal{A}}_{G_A,\psi}(x)^2 := \int_{\ell(x)}^{\ell(Q_0)} \mathfrak{a}_{G_A,\psi}(x,r)^2 \, \frac{dr}{r}.$$

LEMMA 4.56. *We have*

$$\|\mathcal{A}_{G_A,\psi} - \widetilde{\mathcal{A}}_{G_A,\psi}\|_{L^2(G_A)}^2 \lesssim \epsilon^2 \ell(Q_0) + \alpha^4 \|A'\|_2^2.$$

Remark that above we denoted $L^2(G_A) = L^2(\mathcal{H}^1|_{G_A})$.

PROOF. To prove the lemma we need to bound the integrals

$$I_1 := \int_{G_A} \int_0^{\ell(x)} \mathfrak{a}_{G_A,\psi}(x,r)^2 \, \frac{dr}{r} \, d\mathcal{H}^1(x), \quad I_2 := \int_{G_A} \int_{\ell(Q_0)}^{\infty} \mathfrak{a}_{G_A,\psi}(x,r)^2 \, \frac{dr}{r} \, d\mathcal{H}^1(x). \tag{4.56}$$

To do so, we consider the square function $\mathfrak{a}_{G_A,\rho}$, introduced in Section 9. We write

$$\begin{aligned} I_1 \lesssim & \int_{G_A} \int_0^{\ell(x)} |\mathfrak{a}_{G_A,\psi}(x,r) - \mathfrak{a}_{G_A,\rho}(x,r)|^2 \, \frac{dr}{r} \, d\mathcal{H}^1(x) \\ & + \int_{G_A} \int_0^{\ell(x)} \mathfrak{a}_{G_A,\rho}(x,r)^2 \, \frac{dr}{r} \, d\mathcal{H}^1(x) =: I_{1,1} + I_{1,2}. \end{aligned}$$

The first term $I_{1,1}$ can be estimated as in the proof of Lemma 4.38, to obtain

$$I_{1,1} \lesssim \|A'\|_\infty^4 \|A'\|_{L^2(\mathbb{R})}^2 \lesssim \alpha^4 \|A'\|_2^2.$$

Let us look at the term $I_{1,2}$. First, recall from Lemma 4.37 that

$$\begin{aligned} I_{1,2} &\approx \int_{\Pi(G_A)} \int_0^{D(p)/50} \mathfrak{a}_{G_A,\rho}((p,A(p)),r)^2 \, \frac{dr}{r} \, dp \\ &= \int_{\Pi(G_A)} \int_0^{D(p)/50} \left| \int_{q\in\mathbb{R}} \varphi_r(q-p) \left(\frac{A(q)-A(p)}{r} \right) dq \right|^2 \frac{dr}{r} \, dp. \end{aligned} \tag{4.57}$$

where $\varphi_r(\cdot) = \frac{1}{r}\varphi(\frac{\cdot}{r})$. We write the last integral as

$$\sum_{i\in I_{\mathsf{W}}} \int_{R_i} \int_0^{D(p)/50} \left| \int_{q\in\mathbb{R}} \varphi_r(q-p) \left(\frac{A(q)-A(p)}{r} \right) dq \right|^2 \frac{dr}{r} \, dp. \tag{4.58}$$

Observe that for $p \in R_i \in \mathsf{W}$, $0 < r \le D(p)/50 \le \ell(R_i)$, and $q \in \operatorname{supp} \varphi_r(\cdot - p) \subset \bar{B}(p, 1.1r)$, we have $q \in 4R_i$. Thus we can restrict the sum in (4.58) to the intervals R_i such that $4R_i \cap 1.2B_0 \neq \varnothing$, taking also into account that $\operatorname{supp} A \subset 1.2B_0$. Notice also that, by Lemma 4.45, these cubes are contained in $C_7 B_0$, for some absolute constant $C_7 > 1$.

To estimate each of the summands in (4.58), let $p \in R_i$ and $q \in \operatorname{supp} \varphi_r(\cdot - p)$. Taylor's theorem gives, with $\xi_{q,p}$ on the line segment between q and p,

$$A(q) = A(p) + A'(p)(q-p) + \frac{A''(\xi_{q,p})}{2}(q-p)^2.$$

Thus we can write the interior most integral in the right hand side of (4.57) as

$$\frac{1}{r} \int \varphi_r(q-p) A'(p)(p-q) dq + \frac{1}{2r} \int \varphi_r(q-p) A''(\xi_{q,p}) |p-q|^2 \, dq.$$

By symmetry we immediately see that the first integral vanishes. Concerning the second integral, for $p \in R_i \in \mathsf{W}$, $0 < r \le D(p)/50 \le \ell(R_i)$, and $q \in B(p, 1.1r)$ we have $\xi_{q,p} \in 15R_i$, and then from Lemma 4.48 we see that

$$\left| \frac{1}{2r} \int \varphi_r(p-q) A''(\xi_{p,q})(q-p)^2 \, dq \right| \lesssim \frac{1}{r} \sup_{15R_i} |A''(\xi_{p,q})| \, r^2 \lesssim \frac{r\,\epsilon}{\ell(R_i)}.$$

Using again that $D(p)/50 \le \ell(R_i)$ we deduce that

$$I_{1,2} \lesssim \sum_{R_i \subset C_7 B_0} \int_{R_i} \int_0^{D(p)/50} \left| \frac{r\,\epsilon}{\ell(R_i)} \right|^2 \frac{dr}{r} \, dp \lesssim \sum_{R_i \subset C_7 B_0} \epsilon^2 \, \ell(R_i) \lesssim \epsilon^2 \, \ell(Q_0).$$

Next we have to estimate the integral I_2 in (4.56). Given $x \in G_A$ and $r \ge \ell(Q_0)$, let $L_{x,r}$ be a line passing through x and parallel to the line minimizing $\beta_{\infty,G_A}(B(x,1.1r))$, and let $H_{x,r}$ be the half plane whose boundary is $L_{x,r}$ lying above $L_{x,r}$. From the definition of $\mathfrak{a}_{G_A,\psi}(x,r)$, it follows that

$$\left|\mathfrak{a}_{G_A,\psi}(x,r)\right| \lesssim \frac{1}{r^2} |(\Omega_{G_A}^+ \Delta H_{x,r}) \cap B(x,1.1r)| \lesssim \beta_{\infty,G_A}(B(x,2r)). \tag{4.59}$$

Taking into account that $\operatorname{supp} A \subset 1.2B_0$ and that $\operatorname{dist}(x, L_0) \lesssim \epsilon\ell(Q_0)$ for every $x \in G_A$, by Lemma 4.50, it follows easily that

$$\beta_{\infty,G_A}(B(x,2r)) \lesssim \frac{\epsilon\,\ell(Q_0)}{\max\big(r, \operatorname{dist}(x,B_0)\big)} \qquad \text{for all } x \in G_A \text{ and } r \geq \ell(Q_0).$$

So we deduce that

$$\begin{aligned}
I_2 &\lesssim \int_{G_A}\int_{\ell(Q_0)}^{\infty}\left(\frac{\epsilon\,\ell(Q_0)}{\max\big(r,\operatorname{dist}(x,B_0)\big)}\right)^2 \frac{dr}{r}\,d\mathcal{H}^1(x)\\
&\lesssim \int_{G_A\cap 2B_0}\int_{\ell(Q_0)}^{\infty}\big(\epsilon\,\ell(Q_0)\big)^2\frac{dr}{r^3}\,d\mathcal{H}^1 + \int_{G_A\setminus 2B_0}\int_{\ell(Q_0)}^{\infty}\frac{\big(\epsilon\,\ell(Q_0)\big)^2}{\operatorname{dist}(x,B_0)^{3/2}\,r^{1/2}}\,\frac{dr}{r}\,d\mathcal{H}^1(x)\\
&\lesssim \epsilon^2\,\ell(Q_0)^2\int_{\ell(Q_0)}^{\infty}\frac{dr}{r^3} + \epsilon^2\,\ell(Q_0)^2\int_{G_A\setminus 2B_0}\frac{1}{\operatorname{dist}(x,B_0)^{3/2}}\,d\mathcal{H}^1(x)\int_{\ell(Q_0)}^{\infty}\frac{dr}{r^{3/2}}\\
&\lesssim \epsilon^2\,\ell(Q_0),
\end{aligned}$$

Gathering the estimates obtained for $I_{1,1}$, $I_{1,2}$, and I_2, the lemma follows. □

Observe that, from (4.54) and the previous lemma, we have that

$$\|A'\|_2^2 \lesssim \|\widetilde{\mathcal{A}}_{G_A,\psi}\|^2_{L^2(G_A)} + \|\mathcal{A}_{G_A,\psi} - \widetilde{\mathcal{A}}_{G_A,\psi}\|^2_{L^2(G_A)} \lesssim \|\widetilde{\mathcal{A}}_{G_A,\psi}\|^2_{L^2(G_A)} + \epsilon^2\ell(Q_0) + \alpha^4\|A'\|_2^2.$$

Hence, for α small enough, this gives

$$\|A'\|_2^2 \lesssim \|\widetilde{\mathcal{A}}_{G_A,\psi}\|^2_{L^2(G_A)} + \epsilon^2\ell(Q_0).$$

If we put together this and (4.55), we obtain

$$\sum_{Q\in \mathsf{BA}} \mu(Q) \lesssim \alpha^{-2}\|\widetilde{\mathcal{A}}_{G_A,\psi}\|^2_{L^2(G_A)} + \epsilon^2\alpha^{-2}\,\ell(Q_0). \tag{4.60}$$

Note that the implicit constant does not depend on k_1.

To estimate $\|\widetilde{\mathcal{A}}_{G_A,\psi}\|^2_{L^2(G_A)}$, we split

$$\begin{aligned}
\|\widetilde{\mathcal{A}}_{G_A,\psi}\|^2_{L^2(G_A)} &= \int_{G_A}\int_{\ell(x)}^{\ell(Q_0)} \mathfrak{a}_{G_A,\psi}(x,r)^2\,\frac{dr}{r}\,d\mathcal{H}^1(x)\\
&= \int_{G_A}\int_{\ell(x)}^{\min(\epsilon^{-1}\ell(x),\ell(Q_0))}\cdots + \int_{G_A}\int_{\min(\epsilon^{-1}\ell(x),\ell(Q_0))}^{\ell(Q_0)}\cdots\,.
\end{aligned} \tag{4.61}$$

Next we estimate each of these integrals separately.

Lemma 4.57. *We have*

$$\int_{G_A}\int_{\ell(x)}^{\min(\epsilon^{-1}\ell(x),\ell(Q_0))} \mathfrak{a}_{G_A,\psi}(x,r)^2\,\frac{dr}{r}\,d\mathcal{H}^1(x) \leq \epsilon^2|\log\epsilon|\ell(Q_0). \tag{4.62}$$

Proof. From Lemma 4.50, it easily follows that $\beta_{\infty,G_A}(B(x,r)) \lesssim \epsilon$ for all $x \in G_A$ and $r > \ell(x)$. Then, arguing as in (4.59), we deduce that

$$\mathfrak{a}_{G_A,\psi}(x,r) \lesssim \epsilon.$$

Thus, for every $x \in G_A$,

$$\int_{\ell(x)}^{\epsilon^{-1}\ell(x)} \mathfrak{a}_{G_A,\psi}(x,r)^2\,\frac{dr}{r} \lesssim \epsilon^2\int_{\ell(x)}^{\epsilon^{-1}\ell(x)}\frac{dr}{r} \approx \epsilon^2|\log\epsilon|,$$

which yields the lemma, taking into account that for the points x at some distance larger than $C\ell(Q_0)$ with C big enough, $\ell(x) > \ell(Q_0)$ and thus

$$\int_{\ell(x)}^{\min(\epsilon^{-1}\ell(x),\ell(Q_0))} \mathfrak{a}_{G_A,\psi}(x,r)^2\,\frac{dr}{r} = 0.$$

□

To estimate the second integral on the right hand side of (4.61) we need to introduce some additional notation. We denote by Π_{G_A} the projection $\mathbb{R}^2 \to G_A$ orthogonal to L_0. We let

$\mathcal{D}_{G_A}$ be the family of "dyadic cubes" on G_A of the form

$$\mathcal{D}_{G_A} = \{\Pi_{G_A}(I) : I \in \mathcal{D}_{L_0}\}.$$

Then we set

$$\mathsf{W}_{G_A} = \{\Pi_{G_A}(I) : I \in \mathsf{W}\}.$$

We claim that, for each $I \in \mathsf{W}_{G_A}$ which intersects $10B_0$, there exists a cube $Q_I \in \mathsf{Tree}\setminus\mathsf{Stop}$ such that

$$C_5^{-1}\ell(Q_I) \leq \ell(I) \leq C_5\ell(Q_I), \text{ and} \tag{4.63}$$

$$\operatorname{dist}(I, Q_I) \leq C_5\ell(I), \tag{4.64}$$

for some absolute constant C_5. Indeed, by Lemma 4.46, there exists a cube $Q_I \in \mathsf{Tree} \setminus \mathsf{Stop}$ such that $\ell(Q_I) \approx \ell(I)$ and $\operatorname{dist}(I, \Pi_{G_A}(Q_I)) \lesssim \ell(I)$, and then, by Lemma 4.51,

$$\operatorname{dist}(Q_I, G_A) \leq \operatorname{dist}(z_{Q_I}, G_A) \lesssim \epsilon d(z_{Q_I}) \leq \epsilon\, \ell(Q_I),$$

where z_{Q_I} is the center of Q_I. So we have

$$\operatorname{dist}(I, Q_I) \lesssim \operatorname{dist}(I, \Pi_{G_A}(Q_I)) + \operatorname{dist}(Q_I, \Pi_{G_A}(Q_I)) + \operatorname{diam}(\Pi_{G_A}(Q_I)) \lesssim \ell(Q_I) \approx \ell(I).$$

It is immediate to check that the claim also holds for all $I \in \mathsf{W}_{G_A}$ intersecting C_6B_0, for any $C_6 > 1$, allowing now C_5 to depend on C_6.

We need the following auxiliary result.

Lemma 4.58. *For each $I \in \mathsf{W}_{G_A}$ intersecting $10B_0$ there exists some function $g_I \in L^\infty(\mu)$, $g_I \geq 0$ supported on $2B_{\widehat{Q_I}}$ (where $\widehat{Q_I}$ is the parent of Q_I) such that*

$$\int g_I \, d\mu = \mathcal{H}^1(I), \tag{4.65}$$

and

$$\sum_{I \in \mathsf{W}_{G_A} : I \cap 10B_0 \neq \varnothing} g_I \lesssim c(\theta). \tag{4.66}$$

The proof of this lemma is essentially the same as the one of Lemma 7.41 in [To14].

We will also need the next geometric lemma.

Lemma 4.59. *Let $C_6 > 1$ be some absolute constant. If ϵ is small enough,*

$$(\Omega^+_{G_A} \Delta \Omega^+_\Gamma) \cap C_6B_0 \subset \bigcup_{J \in \mathsf{W}_{G_A} : J \cap C_6B_0 \neq \varnothing} B(x_J, C\ell(J)) \cap U_{C\epsilon\ell(J)}(G_A),$$

with C depending on C_6. Also, for each $J \in \mathsf{W}_{G_A}$ such that $J \cap C_6B_0 \neq \varnothing$,

$$\left|(\Omega^+_{G_A} \Delta \Omega^+_\Gamma) \cap B(x_J, C\ell(J))\right| \lesssim \epsilon\, \ell(I)^2.$$

Proof. Let $x \in \Gamma \cap C_6B_0$ and let $J \in \mathsf{W}_{G_A}$ such that $\Pi_{G_A}(x) \in J$. Let $Q_J \in \mathsf{Tree} \setminus \mathsf{Stop}$ satisfy the properties (4.63) and (4.64) and let $x' \in Q_J$. Then $\Pi_{G_A}(x') \in 3C_5J$. If $|x - x'| \geq 10C_5\ell(J)$, from the fact that $\beta_{\infty,\Gamma}(Q') \lesssim \epsilon$ whenever $Q_J \subset Q' \subset Q_0$, we deduce that the minimal cube Q' such that $Q' \supset Q_J$ and $x \in 2B_{Q'}$ satisfies $\angle(L_{Q'}, L_0) \gtrsim 1 \gg \alpha$, which cannot happen. Thus, $|x - x'| \leq 10C_5\ell(J)$ and so $x \in B(x_I, C\ell(J))$ for some constant C depending on C_6. Now, from Lemmas 4.50 and 4.51 and the fact that $\beta_{\infty,\Gamma}(C'Q_J) \lesssim \epsilon$ for a large $C' > 0$, it follows easily that $\operatorname{dist}(x, G_A) \lesssim \epsilon\ell(Q_J) \lesssim \epsilon\ell(J)$, which implies that

$$\Gamma \cap C_6B_0 \subset \bigcup_{J \in \mathsf{W}_{G_A} : J \cap C_6B_0 \neq \varnothing} B(x_J, C\ell(J)) \cap U_{C\epsilon\ell(J)}(G_A),$$

By connectivity arguments, using (4.52), we deduce that

$$\Omega^\pm_\Gamma \cap C_6B_0 \subset \Omega^\pm_{G_A} \cup \bigcup_{J \in \mathsf{W}_{G_A} : J \cap C_6B_0 \neq \varnothing} B(x_J, C\ell(J)) \cap U_{C\epsilon\ell(J)}(G_A), \tag{4.67}$$

which implies the first part of the lemma.

The second part of the lemma follows from (4.67) and the fact that $\beta_{\infty,\Gamma}(C'Q_J) \lesssim \epsilon$. □

Now we are ready to deal with the last integral on the right hand of (4.61):

Lemma 4.60. *We have*

$$\int_{G_A}\int_{\min(\epsilon^{-1}\ell(x),\ell(Q_0))}^{\ell(Q_0)} \mathfrak{a}_{G_A,\psi}(x,r)^2\,\frac{dr}{r}\,d\mathcal{H}^1(x) \tag{4.68}$$
$$\lesssim \epsilon^2\ell(Q_0)+C(\theta)\iint_0^{\ell(Q_0)} \mathfrak{a}_{\Gamma,\psi}(x,r)^2\,\frac{dr}{r}\,d\mu(x).$$

Proof. Denote by $\mathsf{W}_{G_A,0}$ the family of cubes in W_{G_A} which intersect $10B_0$. Since $\epsilon^{-1}\ell(x)\geq \ell(Q)$ for $x\in I\in \mathsf{W}_{G_A}\setminus \mathsf{W}_{G_A,0}$, we have

$$\int_{G_A}\int_{\min(\epsilon^{-1}\ell(x),\ell(Q_0))}^{\ell(Q_0)} \mathfrak{a}_{G_A,\psi}(x,r)^2\,\frac{dr}{r}\,d\mathcal{H}^1(x)$$
$$= \sum_{I\in \mathsf{W}_{G_A,0}} \int_I\int_{\min(\epsilon^{-1}\ell(x),\ell(Q_0))}^{\ell(Q_0)} \mathfrak{a}_{G_A,\psi}(x,r)^2\,\frac{dr}{r}\,d\mathcal{H}^1(x). \tag{4.69}$$

Given $x\in I\in \mathsf{W}_{G_A,0}$, we consider an arbitrary point $x'\in 2B_{\widehat{Q_I}}$. Then we write

$$|\mathfrak{a}_{G_A,\psi}(x,r)-\mathfrak{a}_{\Gamma,\psi}(x',r)|\leq |\mathfrak{a}_{G_A,\psi}(x,r)-\mathfrak{a}_{G_A,\psi}(x',r)|+|\mathfrak{a}_{G_A,\psi}(x',r)-\mathfrak{a}_{\Gamma,\psi}(x',r)|\,. \tag{4.70}$$

Regarding the first sum on the right hand side, using the fact that $r\geq \epsilon^{-1}\ell(x)\approx \epsilon^{-1}\ell(I)\gg \ell(I)$ and taking into account that $|x-x'|\leq \operatorname{dist}(I,2B_{\widehat{Q_I}})\lesssim \ell(I)$ by (4.64), we get

$$|\mathfrak{a}_{G_A,\psi}(x,r)-\mathfrak{a}_{G_A,\psi}(x',r)|\leq r^{-2}\int_{\Omega^+_{G_A}}\left|\psi\left(\frac{x-y}{r}\right)-\psi\left(\frac{x'-y}{r}\right)\right|dy$$
$$\lesssim \|\nabla\psi\|_\infty \int_{B(x,2r)}\frac{|x-x'|}{r^3}\,dy\lesssim \frac{\ell(I)}{r},$$

Next we deal with the last term in (4.70):

$$|\mathfrak{a}_{G_A,\psi}(x',r)-\mathfrak{a}_{\Gamma,\psi}(x',r)|=r^{-2}\left|\int_{\Omega^+_{G_A}}\psi\left(\frac{x'-y}{r}\right)dy-\int_{\Omega^+_{\Gamma}}\psi\left(\frac{x'-y}{r}\right)dy\right|$$
$$\lesssim r^{-2}\left|(\Omega^+_{G_A}\Delta\Omega^+_{\Gamma})\cap B(x,2r)\right|.$$

Notice next that, insofar as $I\in \mathsf{W}_{G_Q}$, there is an absolute constant such that if x_I is the center of I, then if $J\in \mathsf{W}_{G_Q}$ satisfies $J\cap B(x,2r)\neq\varnothing$, then $J\subset B(x,Cr)$. But then, Lemma 4.59 ensures that, for some constant $C>0$ big enough

$$\left|(\Omega^+_{G_A}\Delta\Omega^+_{\Gamma})\cap B(x,2r)\right|\leq \sum_{J\in \mathsf{W}_{G_A}:J\cap B(x,2r)\neq\varnothing}\left|(\Omega^+_{G_A}\Delta\Omega^+_{\Gamma})\cap B(x_J,C\ell(J))\right|$$
$$\lesssim \sum_{J\in \mathsf{W}_{G_A}:J\subset B(x_I,Cr)}\epsilon\,\ell(J)^2.$$

From the last estimates we derive

$$|\mathfrak{a}_{G_A,\psi}(x,r)-\mathfrak{a}_{\Gamma,\psi}(x',r)|^2\lesssim \frac{\ell(I)^2}{r^2}+\epsilon^2\Bigg(\sum_{J\in \mathsf{W}_{G_A}:J\subset B(x_I,Cr)}\frac{\ell(J)^2}{r^2}\Bigg)^2.$$

Since

$$\sum_{J\in \mathsf{W}_{G_A}:J\subset B(x_I,Cr)}\ell(J)^2\lesssim r\sum_{J\in \mathsf{W}_{G_A}:J\subset B(x_I,Cr)}\ell(J)\lesssim r^2,$$

we deduce that

$$|\mathfrak{a}_{G_A,\psi}(x,r)-\mathfrak{a}_{\Gamma,\psi}(x',r)|^2\lesssim \frac{\ell(I)^2}{r^2}+\epsilon^2\sum_{J\in \mathsf{W}_{G_A}:J\subset B(x_I,Cr)}\frac{\ell(J)^2}{r^2}.$$

Since this holds for all $x'\in 2B_{\widehat{Q_I}}$,

$$\mathfrak{a}_{G_A,\psi}(x,r)^2\lesssim \inf_{x'\in 2B_{\widehat{Q_I}}}\mathfrak{a}_{\Gamma,\psi}(x',r)^2+\frac{\ell(I)^2}{r^2}+\epsilon^2\sum_{J\in \mathsf{W}_{G_A}:J\subset B(x_I,Cr)}\frac{\ell(J)^2}{r^2}$$

for all $x\in I\in \mathsf{W}_{G_A,0}$.

We use (4.69) and the last inequality to estimate the integral on the left side of (4.68):

$$\begin{aligned}
\int_{G_A}\int_{\min(\epsilon^{-1}\ell(x),\ell(Q_0))}^{\ell(Q_0)} & \mathfrak{a}_{G_A,\psi}(x,r)^2\,\frac{dr}{r}\,d\mathcal{H}^1(x) \\
&\lesssim \sum_{I\in\mathsf{W}_{G_A,0}}\int_I\int_{\min(c\epsilon^{-1}\ell(I),\ell(Q_0))}^{\ell(Q_0)} \inf_{x'\in 2B_{\widehat{Q_I}}}\mathfrak{a}_{\Gamma,\psi}(x',r)^2\,\frac{dr}{r}\,d\mathcal{H}^1(x) \\
&\quad+\sum_{I\in\mathsf{W}_{G_A,0}}\int_I\int_{\min(c\epsilon^{-1}\ell(I),\ell(Q_0))}^{\ell(Q_0)} \frac{\ell(I)^2}{r^2}\,\frac{dr}{r}\,d\mathcal{H}^1(x) \\
&\quad+\sum_{I\in\mathsf{W}_{G_A,0}}\int_I\int_{\min(c\epsilon^{-1}\ell(I),\ell(Q_0))}^{\ell(Q_0)} \epsilon^2\sum_{\substack{J\in\mathsf{W}_{G_A}:\\ J\subset B(x_I,Cr)}}\frac{\ell(J)^2}{r^2}\,\frac{dr}{r}\,d\mathcal{H}^1(x) \\
&=: T_1+T_2+T_3.
\end{aligned}$$

First we estimate T_2:

$$T_2\lesssim\sum_{I\in\mathsf{W}_{G_A,0}}\ell(I)\int_{c\epsilon^{-1}\ell(I)}^{\infty}\frac{\ell(I)^2}{r^3}\,dr\lesssim\epsilon^2\sum_{I\in\mathsf{W}_{G_A,0}}\ell(I)\lesssim\epsilon^2\,\ell(Q_0).$$

To we deal with T_3, given $I,J\in\mathsf{W}_{G_A}$ with $J\subset B(x_I,Cr)$, $r>c\epsilon^{-1}\ell(I)$, we denote

$$D(I,J)=\ell(I)+\ell(J)+\operatorname{dist}(I,J).$$

Observe that the preceding conditions on I,J,r imply that $r\gtrsim D(I,J)$. Then, using Fubini we get

$$\begin{aligned}
T_3&\lesssim\sum_{I\in\mathsf{W}_{G_A,0}}\ell(I)\int_{\min(c\epsilon^{-1}\ell(I),\ell(Q_0))}^{\ell(Q_0)}\epsilon^2\sum_{\substack{J\in\mathsf{W}_{G_A}:\\ J\subset B(x_I,Cr)}}\frac{\ell(J)^2}{r^3}\,dr \\
&\lesssim\epsilon^2\sum_{J\in\mathsf{W}_{G_A}:J\subset CB_0}\ell(J)^2\sum_{I\in\mathsf{W}_{G_A}}\ell(I)\int_{r>c\,D(I,J)}\frac{dr}{r^3} \\
&\lesssim\epsilon^2\sum_{J\in\mathsf{W}_{G_A}:J\subset CB_0}\ell(J)^2\sum_{I\in\mathsf{W}_{G_A}}\frac{\ell(I)}{D(I,J)^2}.
\end{aligned}$$

To bound the last sum, we split it as follows:

$$\begin{aligned}
\sum_{I\in\mathsf{W}_{G_A}}\frac{\ell(I)}{D(I,J)^2}&=\sum_{k\geq0}\sum_{\substack{I\in\mathsf{W}_{G_A}:\\ 2^k\ell(J)\leq D(I,J)\leq 2^{k+1}\ell(J)}}\frac{\ell(I)}{D(I,J)^2} \\
&\lesssim\sum_{k\geq0}\frac{1}{2^{2k}\ell(J)^2}\sum_{\substack{I\in\mathsf{W}_{G_A}:\\ I\subset B(x_J,C2^k\ell(J))}}\ell(I)\lesssim\sum_{k\geq0}\frac{1}{2^k\ell(J)}\approx\frac{1}{\ell(J)}.
\end{aligned}$$

So we have

$$T_3\lesssim\epsilon^2\sum_{J\in\mathsf{W}_{G_A}:J\subset CB_0}\ell(J)\lesssim\epsilon^2\ell(Q_0).$$

Finally we will estimate the tern T_1. To this end, we consider the functions g_I constructed in Lemma 4.58. It is clear that

$$T_1=\sum_{I\in\mathsf{W}_{G_A,0}}\iint_{\min(c\epsilon^{-1}\ell(I),\ell(Q_0))}^{\ell(Q_0)}\inf_{x'\in 2B_{\widehat{Q_I}}}\mathfrak{a}_{\Gamma,\psi}(x',r)^2\,\frac{dr}{r}\,g_I(x)\,d\mu(x).$$

Observe now that, for each $x\in 2B_{\widehat{Q_I}}$,

$$\inf_{x'\in 2B_{\widehat{Q_I}}}\mathfrak{a}_{\Gamma,\psi}(x',r)\leq\fint_{2B_{\widehat{Q_I}}}\mathfrak{a}_{\Gamma,\psi}(x',r)\,d\mu(x')\leq\frac{\mu(6B_{\widehat{Q_I}})}{\mu(2B_{\widehat{Q_I}})}\,\widetilde{M}_\mu\mathfrak{a}_{\Gamma,\psi}(\cdot,r)(x),$$

where $\widetilde{M}_\mu$ is the maximal operator defined by

$$\widetilde{M}_\mu f(x) = \sup_{B \ni x} \frac{1}{\mu(3B)} \int_B |f|\, d\mu.$$

Since $\frac{\mu(6B_{\widehat{Q_I}})}{\mu(2B_{\widehat{Q_I}})} \lesssim \theta^{-1}$, using Fubini and Lemma 4.58, we can write

$$\begin{aligned}
T_1 &\lesssim \theta^{-1} \sum_{I \in \mathsf{W}_{G_A,0}} \iint_{\min(c\epsilon^{-1}\ell(I), \ell(Q_0))}^{\ell(Q_0)} \widetilde{M}_\mu \mathfrak{a}_{\Gamma,\psi}(\cdot, r)(x)^2 \, \frac{dr}{r}\, g_I(x)\, d\mu(x) \\
&\leq \theta^{-1} \int_0^{\ell(Q_0)} \int \widetilde{M}_\mu \mathfrak{a}_{\Gamma,\psi}(\cdot, r)(x)^2 \sum_{I \in \mathsf{W}_{G_A,0}} g_I(x)\, d\mu(x)\, \frac{dr}{r} \\
&\lesssim c(\theta) \int_0^{\ell(Q_0)} \int \widetilde{M}_\mu \mathfrak{a}_{\Gamma,\psi}(\cdot, r)(x)^2 \, d\mu(x)\, \frac{dr}{r}
\end{aligned}$$

Using that $\widetilde{M}_\mu$ is bounded in $L^2(\mu)$ (see Theorem 9.32 in [To14], for example), we derive

$$T_1 \lesssim c(\theta) \int_0^{\ell(Q_0)} \int \mathfrak{a}_{\Gamma,\psi}(x, r)^2 \, d\mu(x)\, \frac{dr}{r}.$$

Gathering the estimates obtained for the terms T_1, T_2, and T_3, the lemma follows. $\square$

Proof of Lemma 4.54. By (4.60) and Lemmas 4.57, 4.60, we get

$$\begin{aligned}
&\sum_{Q \in \mathsf{BA}} \mu(Q) \lesssim \alpha^{-2} \epsilon^2\, \ell(Q_0) + \alpha^{-2} \|\widetilde{\mathcal{A}}_{G_A,\psi}\|^2_{L^2(G_A)} \\
&\lesssim \alpha^{-2}\epsilon^2\, \ell(Q_0) + \alpha^{-2} \left(\epsilon^2 |\log \epsilon| \ell(Q_0) + \epsilon \ell(Q_0) + C(\theta) \iint_0^{\ell(Q_0)} \mathfrak{a}_{\Gamma,\psi}(x, r)^2 \, \frac{dr}{r}\, d\mu(x) \right) \\
&\lesssim \alpha^{-2}\epsilon^2 |\log \epsilon|\, \ell(Q_0) + C(\theta, \alpha) \iint_0^{\ell(Q_0)} \mathfrak{a}_{\Gamma,\psi}(x, r)^2 \, \frac{dr}{r}\, d\mu(x) \leq \epsilon^{1/2} \mu(Q_0),
\end{aligned}$$

for $\epsilon = \epsilon(\alpha, \theta)$ small enough. This yields the desired conclusion. $\square$

12.3. Proof of the Main Lemma 4.7. By Lemmas 4.53 and 4.54, if θ is chosen small enough and then ϵ also small enough (depending on α and θ), then

$$\mu(\mathsf{BA} \cup \mathsf{LD}) \leq \frac{1}{2}\, \mu(Q_0).$$

This implies that

$$\mu(\mathsf{Out}) = \mu(\mathsf{Stop} \setminus (\mathsf{BA} \cup \mathsf{LD})) \geq \frac{1}{2}\, \mu(Q_0).$$

From Lemma 4.51 we deduce that

$$\bigcup_{Q \in \mathsf{Out}} Q \subset U_{A_0^{-k_1} \ell(Q_0)}(G_A).$$

Therefore, writing $G_A^{k_1}$ in place of G_A,

$$\mu\Big(U_{A_0^{-k_1}\ell(Q_0)}(G_A^{k_1})\Big) \geq \frac{1}{2}\, \mu(Q_0). \tag{4.71}$$

Taking the limit as $k_1 \to \infty$, there is a subsequence such that the Lipschitz graphs $G_A^{k_1}$ converge in Hausdorff distance to another Lipschitz graph Λ. From the condition (4.71) and the regularity properties of Radon measures it follows easily that

$$\mu(\Lambda) \geq \frac{1}{2}\, \mu(Q_0).$$

CHAPTER 5

Appendix

1. Notation

In this subsection we gather some basic facts and notation that will be used throughout the rest of this thesis.

Throughout this thesis, d and n denote two integers with $0 < d < n$; n will be used to denote the dimension of the ambient space $\mathbb{R}^n$, whilst d will denote the intrinsic dimension (in what precise sense will be specified case by case) of the set or measure under investigation.

We will denote by $C, c > 0$ absolute constants that may change from line to line. We will often use the symbol $A \lesssim B$ to mean that $A \leq CB$. The symbol $A \gtrsim B$ is just another way of writing $B \lesssim A$. The symbol $A \approx B$ means that both $A \lesssim B$ and $B \lesssim A$. If a constant is allowed to depend on a given parameter, the parameter dependence will be described in parenthesis or a subscript; for example, $C_{\epsilon,\varkappa}$ and $C(\epsilon, \varkappa)$ both denote a constant that may depend on parameters ϵ and $\varkappa$. Then $A \lesssim_{\epsilon,\varkappa} B$ means that $A \leq C(\epsilon, \varkappa)B$.

For sets $A, B \subset \mathbb{R}^n$, we let

$$\operatorname{dist}(A, B) := \inf_{a\in A, b\in B} |a - b|.$$

For a point $x \in \mathbb{R}^n$ and a subset $A \subset \mathbb{R}^n$,

$$\operatorname{dist}(x, A) := \operatorname{dist}(\{x\}, A) = \inf_{a\in A} \operatorname{dist}(x, a).$$

We shall write $A \ll B$ to mean that "A is much smaller than B", namely that $A \leq cB$ for a sufficiently small absolute constant $c > 0$.

1.1. Balls and annuli. Balls $B(x, r)$ are assumed to be open. Also, when we say that a set $B \subset \mathbb{R}^2$ is a ball, we mean an open ball, unless otherwise stated. We denote by $r(B)$ its radius.

The notation $A(x, r, R)$ stands for an open annulus centered at x with inner radius r and outer radius R.

1.2. Hausdorff measure and content. Let $A \in \mathbb{R}^n$ and $0 < \delta \leq \infty$. Set

$$\mathcal{H}^d_\delta(A) := \inf \left\{ \sum (\operatorname{diam}(A_i))^d \,|\, A \subset \cup_i A_i \text{ and } \operatorname{diam}(A_i) \leq \delta \right\}. \tag{5.1}$$

The d-dimensional Hausdorff measure of A is then defined by

$$\mathcal{H}^d(A) := \lim_{\delta\to 0} \mathcal{H}^d_\delta(A).$$

The d-dimensional Hausdorff content of A is as in (5.1) with $\delta = \infty$. For properties of Hausdorff measures and content, see [**Mat95**], Chapter 4.

1.3. Rectifiability. This is the direct analogue of the definitions given in Chapter 1 for general dimensions n and d (there we had $n = 2$ and $d = 1$).

Definition 5.1. A set $A \subset \mathbb{R}^n$ is called *d-rectifiable* if there are Lipschitz functions $f_i : \mathbb{R}^d \to \mathbb{R}^n$, $i = 1, 2, ...$ such that

$$\mathcal{H}^d \left(A \setminus \bigcup_i f_i(\mathbb{R}^d) \right) = 0.$$

On the other hand, a set A is called *purely d-unrectifiable* if $\mathcal{H}^d(A \cap F) = 0$ whenever F is the image of a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}^n$.

The book [**Mat95**] contains a wealth of information on rectifiable sets.

2. 'Dyadic cubes' for general sets

In this section, we recall two systems of 'dyadic cubes' for general sets. They are essentially the same, except in the parameters and constant. Chapter 2 and Chapter 3 will be based on the so called Christ-David cubes, while Chapter 4 will be based on the David-Mattila cubes. This is because the original papers on which the different chapters are based adopted these systems.

2.1. Christ-David cubes. This version of dyadic cubes for metric spaces was first introduced by David [**Dav88**] but then generalised in [**C90**] and [**HM12**].

THEOREM 5.2. *Let X be a doubling metric space. Let X_k be a nested sequence of maximal λ^k-nets for X where $\lambda < 1/1000$ and let $c_0 = 1/500$. For each $n \in \mathbb{Z}$ there is a collection $\mathcal{D}_k$ of "cubes," which are Borel subsets of X such that the following hold.*

(1) For every integer k, $X = \bigcup_{Q\in\mathcal{D}_k} Q$.
(2) If $Q, Q' \in \mathcal{D} = \bigcup \mathcal{D}_k$ and $Q \cap Q' \neq \emptyset$, then $Q \subseteq Q'$ or $Q' \subseteq Q$.
(3) For $Q \in \mathcal{D}$, let $k(Q)$ be the unique integer so that $Q \in \mathcal{D}_k$ and set $\ell(Q) = 5\lambda^{k(Q)}$. Then there is $z_Q \in X_k$ so that

$$B(z_Q, c_0\ell(Q)) \subseteq Q \subseteq B(z_Q, \ell(Q)) \tag{5.2}$$

and $X_k = \{z_Q \mid Q \in \mathcal{D}_k\}$.

We set some notation. For a cube $Q \in \mathcal{D}_k$, we put

$$\mathsf{Child}(Q) := \{Q' \in \mathcal{D}_{k+1} \,|\, Q' \subset Q\}\,. \tag{5.3}$$

Moreover, for a cube $Q \in \mathcal{D}$, we put

$$B(Q) := B(z_Q, c_0\ell(Q)) \text{ and } B_Q := B(z_Q, \ell(Q)). \tag{5.4}$$

In our applications of Theorem 5.2 in Chapter 2 and 3 we will only use the dyadic partition of the set (or space), without any reference to a measure.

2.2. David-Mattila cubes. The following system was constructed in [**DM**]. We rematk that we will only need very few properties of the David-Mattila lattice.

THEOREM 5.3. *Let μ be a Radon measure on $\mathbb{R}^n$, set $E = \operatorname{supp}(\mu)$. For any constants $C_0 > 1$, $A_0 > 5000C_0$, there exists a sequence of partitions of E into Borel subsets Q, $Q \in \mathcal{D}_k$, with the following properties.*

(a) For each integer $k \geq 0$, E is the disjoint union of 'cubes' Q, $Q \in \mathcal{D}_k$; if $k > l$, $Q \in \mathcal{D}_l$ and $P \in \mathcal{D}_k$, then wither $Q \cap P = \varnothing$ or $P \subset Q$.
(b) The general position of the cubes Q can be described as follows. For each $k \geq 0$ and each cube $Q \in \mathcal{D}_k$, there is a ball

$$B(Q) := B(z_Q, r(Q)),$$

such that

$$z_Q \in Q,\ A_0^{-k} \leq r(Q) \leq C_0 A_0^{-k}$$
$$E \cap B(Q) \subset Q \subset E \cap 28B(Q) = B(z_Q, 28r(Q)),$$

and the balls $5B(Q)$, for $Q \in \mathcal{D}_k$, are disjoint.

The dyadic lattice constructed by David and Mattila has many other remarkable properties which we do not list because we will not need them.

Let us fix some notation. Given $Q \in \mathcal{D}_k$, we refer to the number k as the *generation* of Q; also, we put $\mathcal{D}(Q)$ to mean the family of cubes from $\mathcal{D}$ which also are contained in Q. We set

$$\ell(Q) := 56C_0A_0^{-k}.$$

We call $\ell(Q)$ the side length of Q. We denote by $\mathsf{Child}(Q)$ the cubes from $\mathcal{D}_{k+1}$ which are contained in Q. We also put

$$B_Q := 28B(Q) = B(z_Q, 28r(Q)).$$

2.3. Stopping time regions or trees. We define *stopping time regions* or *trees* as follows; these definitions are independent of the choice of dyadic system.

DEFINITION 5.4. A collection $\mathcal{F} \subset \mathcal{D}$ is a *stopping-time region* or *tree* if the following hold:

(1) There is a cube $Q(\mathcal{F}) \in \mathcal{F}$ that contains every cube in $\mathcal{F}$.
(2) If $Q \in \mathcal{F}$, $R \in \mathcal{F}$, and $Q \subset R \subset Q(\mathcal{F})$, then $R \in \mathcal{F}$.
(3) $Q \in \mathcal{F}$ and there is $Q' \in \mathsf{Child}(Q) \backslash \mathcal{F}$, then $\mathsf{Child}(Q) \subset \mathcal{F}^c$.

Bibliography

[Azz] J. Azzam, Harmonic Measure and the Analyst's Traveling Salesman Theorem *preprint*. 24

[AT15] Azzam J, Tolsa X. Characterization of n-rectifiability in terms of Jones square function: Part II. Geometric and Functional Analysis. 2015 Oct 1;25(5):1371-412. 16

[AHM+17] J. Azzam, S. Hofmann, J.M. Martell, K. Nyström, and T. Toro. A new characterization of chord-arc domains. *J. Eur. Math. Soc. (JEMS)*, 19(4):967–981, 2017.

[AS18] Azzam J, Schul R. An analysts traveling salesman theorem for sets of dimension larger than one. Mathematische Annalen. 2018 Apr 1;370(3-4):1389-476. 17, 19, 23, 38, 45, 48, 49, 78

[AV19] Azzam J, Villa M. Quantitative Comparisons of Multiscale Geometric Properties. arXiv preprint arXiv:1905.00101. 2019 Apr 30.

[B] C. J. Bishop, *Some questions concerning harmonic measure*, Partial differential equations with minimal smoothness and applications (Chicago, IL, 1990), IMA Vol. Math. Appl., vol. 42, Springer, New York, 1992, pp. 89–97. 25

[BL] M. Badger and S. Lewis, *Local set approximation: Mattila-Vuorinen type sets, Reifenberg type sets, and tangent sets.* Forum Math. Sigma 3 (2015), e24, 63 pp. 85

[BJ94] C.J. Bishop and P.W. Jones. *Harmonic measure, L^2 estimates and the Schwarzian derivative.* J. Ana. Math., VoL 62 (1994),77–112. 92
85, 86

[BJ97] Bishop CJ, Jones PW. Wiggly sets and limit sets. Arkiv for Matematik. 1997 Oct 1;35(2):201-24. 47, 78

[Ca77] Calderòn A.P. Cauchy integrals on Lipschitz curves and related operators. Proceedings of the National Academy of Sciences. 1977 Apr 1;74(4):1324-7. 13

[CW77] Coifman, Ronald R., Weiss, Guido. Extensions of Hardy spaces and their use in analysis. Bull. Amer. Math. Soc. 83 (1977), no. 4, 569–645. 14

[C90] Christ M. A T(b) Theorem with remarks on analytic capacity and the Cauchy integral. *Colloquium Mathematicum*, 1990. 13, 128

[C90b] Christ FM. *Lectures on singular integral operators.* CBMS Regional Conf. Ser. in Math, vol. 77 (1990) American Mathematical Soc., Providence, RI. 13

[Dav88] G. David. Morceaux de graphes lipschitziens et intégrales singulières sur une surface. *Rev. Mat. Iberoamericana*, 4(1):73–114, 1988. 50, 128

[Dav98] G. David. Unrectifiable 1-sets have vanishing analytic capacity. Rev. Mat. Iberoamericana, 14(2):369479, 1998. 14

[Dav04] David G. Hausdorff dimension of uniformly non flat sets with topology. Publicacions matematiques. 2004 Jan 1:187-225. 19, 47, 48, 49, 50, 55, 56, 60, 65, 67, 69, 70, 71, 72, 78

[Dav06] David G. Wavelets and singular integrals on curves and surfaces. Springer; 2006 Nov 14. 37

[DM] G. David and P. Mattila. *Removable sets for Lipschitz harmonic functions in the plane.* Rev. Mat. Iberoamericana 16(1) (2000), 137–215. 128

[DS91] David G, Semmes S. Singular integrals and rectifiable sets in $\mathbb{R}^n$: Au-dela des graphes lipschitziens. Société mathématique de France; 1991. 14, 24, 29, 35, 37, 40, 53, 85, 86, 89, 116, 117

[DS93] David G, Semmes S. *Analysis of and on uniformly rectifiable sets.* American Mathematical Soc.; 1993. 13, 14, 16, 24, 35, 37, 42, 43, 44, 49, 75

[DS00] David G, Semmes S. Uniform rectifiability and quasiminimizing sets of arbitrary codimension. American Mathematical Soc.; 2000. 47, 51, 56, 57, 65, 68, 70, 72, 73

[DS17] G. C. David and R. Schul. The analyst's traveling salesman theorem in graph inverse limits. *Ann. Acad. Sci. Fenn. Math.*, 42(2):649–692, 2017.

[DS19] G. C. David and R. Schul. A sharp necessary condition for rectifiable curves in metric spaces. *arXiv preprint arXiv:1902.04030*, 2019.

[DT12] David G, Toro T. Reifenberg parameterizations for sets with holes. American Mathematical Soc.; 2012.

[D85] Dorronsoro JR. A characterization of potential spaces. Proceedings of the American Mathematical Society. 1985;95(1):21-31. 15

[DMSV16] De Lellis C, Marchese A, Spadaro E, Valtorta D. Rectifiability and upper Minkowski bounds for singularities of harmonic Q-valued maps. arXiv preprint arXiv:1612.01813. 2016 Dec 6. 16

[EE19] Edelen N, Engelstein M. Quantitative stratification for some free-boundary problems. Transactions of the American Mathematical Society. 2019;371(3):2043-72. 16

[Fa90] Fang X. The Cauchy Integral of Calderón and Analytic Capacity. PhD dissertation (1990). 17

[Fe69] Federer H. *Geometric measure theory.* Springer, 1969. 91

[FFP07] F. Ferrari, B. Franchi, and H. Pajot. The geometric traveling salesman problem in the Heisenberg group. *Rev. Mat. Iberoam.*, 23(2):437–480, 2007.

[GM] J. B. Garnett and D. E. Marshall. *Harmonic Measure*. New Mathematical Monographs **2**. Cambridge University Press, Cambridge, 2005. 79, 85

[H05] Heinonen J. *Lectures on Lipschitz analysis*. University of Jyväskylä; 2005. 57

[HLMN17] S. Hofmann, P. Le, J.M. Martell, and K. Nyström. The weak-A_∞ property of harmonic and p-harmonic measures implies uniform rectifiability. *Anal. PDE*, 10(3):513–558, 2017. 16

[HM18] S. Hofmann and J. M. Martell. On quantitative absolute continuity of harmonic measure and big piece approximation by chord-arc domains. *arXiv preprint arXiv:1712.03696*, 2018.

[HM14] S. Hofmann and J. M. Martell. Uniform rectifiability and harmonic measure I: Uniform rectifiability implies Poisson kernels in L^p. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(3):577–654, 2014.

[HM12] T. Hytönen and H. Martikainen. Non-homogeneous Tb theorem and random dyadic cubes on metric measure spaces. *J. Geom. Anal.*, 22(4):1071–1107, 2012. 128

[JTV] B. Jaye, X. Tolsa and M. Villa. A proof of the Carleson ε-conjecture. Arxiv preprint. arXiv:1909.08581v1 85, 86

[JK] D. S. Jerison and C. E. Kenig. *Boundary behavior of harmonic functions in nontangentially accessible domains*, Adv. in Math. 46 (1982), no. 1, 80–147. 87

[J90] Jones P.W. Rectifiable sets and the traveling salesman problem. Inventiones Mathematicae. 1990 Dec 1;102(1):1-5. 17, 86

[J91] Jones P.W. The traveling salesman problem and harmonic analysis. Publicacions Matemtiques. 1991 Jan 1;35(1):259-67.

[KT] C. Kenig and T. Toro. *Poisson kernel characterization of Reifenberg flat chord arc domains*, Ann. Sci. Ecole Norm. Sup. (4) 36 (2003), no.3, 323–401. 96

[Lé] J.C. Léger, *Menger curvature and rectifiability,* Ann. of Math. 149 (1999), 831-869. 86, 89, 116

[Mat96] Mattila P. On the analytic capacity and curvature of some Cantor sets with non-σ-finite length. Publ. Mat., 40(1):195204, 1996. 14

[Mat95] Mattila P. *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability.* Cambridge university press; 1999 Feb 25. 13, 26, 78, 81, 90, 91, 127

[MMV97] Mattila P, Melnikov M.S., Verdera J. The Cauchy integral, analytic capacity, and uniform rectifiability. Annals of Mathematics. 1996 Jul 1;144(1):127-36. 14

[Mel95] Melnikov M.S. Analytic capacity: a discrete approach and the curvature of measure. Mat. Sb., 186(6):5776, 1995. 14

[NTV14] F. Nazarov, X. Tolsa, and A. Volberg. On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: the case of codimension 1. *Acta Math.*, 213(2):237–321, 2014. 16

[Oki] Okikiolu K. Characterization of subsets of rectifiable curves in Rn. Journal of the London Mathematical Society. 1992 Oct 1;2(2):336-48. 17

[S] Semmes SW. Finding structure in sets with little smoothness. InProceedings of the International Congress of Mathematicians 1995 (pp. 875-885). Birkhauser, Basel. 47

[Sch07] Schul R. Subsets of rectifiable curves in Hilbert space-the analyst's TSP. Journal d'Analyse Mathematique. 2007 Dec 1;103(1):331-75. 17

[Si14] Simon L. *Introduction to geometric measure theory. Lectures on geometric measure theory* (Canberra, 1983), revised edition. 2014. 91

[SM93] Stein EM, Murphy TS. Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton University Press; 1993. 15

[To03] X. Tolsa. Painlevés problem and the semiadditivity of analytic capacity. Acta Math., 190(1):105149, 2003. 14

[To14] X. Tolsa. *Analytic capacity, the Cauchy transform, and non-homogeneous Calderón-Zygmund theory.* Birkhäuser, 2014. 13, 97, 117, 118, 119, 120, 123, 126

[To15] X Tolsa. Uniform measures and uniform rectifiability. *J. London Math. Soc.*, 92(1):1-8, 2015. 37

[V19b] Villa M. Sets with topology, the Analyst's TST, and applications. arXiv preprint arXiv:1908.10289. 2019 Aug 27. 49